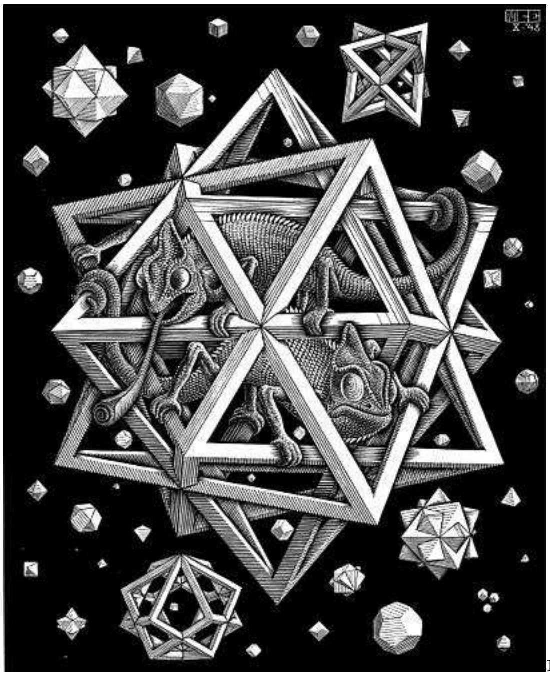
# Platonic solids, binary polyhedral groups, Kleinian singularities and Lie algebras of type $A$, $D$, $E$.

Master's Thesis of Joris van Hoboken,
under supervision of Prof.dr. E. M. Opdam



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# Contents

# 1 Introduction

This thesis is an attempt to show an astonishing relation between basic objects from different fields in mathematics. Most peculiarly it turns out that their classification is 'the same': the $ADE$ classification. Altogether these objects and the connections between them form a coherent web.

The connections are accomplished by direct mathematical constructions leading to bijections between these classes of objects. These constructions however do not always explain or give satisfactory intuition why these classifications are, or to say it better, why they should be related in this way. Therefore the deeper reason remains mysterious and when discovered will have to be of great depth. This gives a high motivation to look for new concepts and it shows that simple and since long understood mathematics can still raise very interesting questions, show paths for new research and give a glance at the mystery of mathematics. In my opinion to be aware of a certain truth without having its reason is fundamental to the practise of mathematics.

In this thesis I will demonstrate the following correspondences, in this order:

| | | |
|---|---|---|
| Platonic solids | $\leftrightarrow$ | binary polyhedral groups |
| binary polyhedral groups | $\leftrightarrow$ | Kleinian singularities |
| Kleinian singularities | $\leftrightarrow$ | Dynkin diagrams |
| representations of bin. pol. groups | $\leftrightarrow$ | affine Dynkin diagrams |
| representations of bin. pol. groups | $\leftrightarrow$ | Kleinian singularities |
| Lie algebras of type A, D, E | $\leftrightarrow$ | Kleinian singularities |

The first connection was found in the end of the nineteenth century. It was Felix Klein who studied the theory of the regular polyhedra and their symmetries, in his book on the icosahedron, see [18]. He found the relation between the possible finite subgroups of the special unitary group $SU_2(\mathbb{C})$ and the platonic solids. For these groups he computed a fundamental set of invariant polynomials which he found to be subject to one essential relation. He used these results, which we will discuss in the first sections, for his theory of equations of fifth degree.

The results of Klein on the invariant theory of the binary polyhedral groups, $\Gamma \subset SU_2(\mathbb{C})$, were a starting point for later developments. In the 1920's du Val obtained a resolution of the Kleinian singularities $\mathbb{C}^2/\Gamma$; the quotient singularities associated with the binary polyhedral groups. He was able to give a characterization that made them a particular class of singularities: "Isolated singularities of surfaces which do not affect the conditions of adjunction". After this other characterizations of these singularities have been found, for instance by Arnold. We will show that with the use of a minimal resolution $\tilde{S} \to \mathbb{C}^2/\Gamma$ with exceptional set $E$ they can be classified by the Dynkin diagrams $\Delta$ of type $A_r$, $D_r$, $E_6$, $E_7$ and $E_8$.

Let us summarize this in a picture:

$$\begin{array}{ccc} & & \mathbb{C}^2 \\ & & \downarrow \pi \\ E = \bigcup E_i \subset \tilde{S} & \to & \mathbb{C}^2/\Gamma \end{array}$$

These Dynkin diagrams classify another special class of objects, the simple Lie algebras or Lie groups which have root systems with all roots the same length.

In this field of Lie theory and algebraic group theory great progress had been made; the classification of semisimple Lie algebras by Dynkin diagrams, the theory of reflection groups and Weyl groups and their representations, the study of conjugacy classes in algebraic groups. With this theory various other surprising connections with binary polyhedral groups and Kleinian singularities were found.

As stated above the Dynkin diagrams of type $A$, $D$ and $E$ arise naturally in the resolution of the quotient singularities of binary polyhedral groups. Then in 1980 McKay found there is a direct connection between their affine versions and the irreducible representations of these groups and soon after this it was shown that also the geometry of the resolution of the quotient singularity and the irreducible representations are directly related.

After his work on singularity theory and the work of Kostant, Steinberg, and Springer on the adjoint quotients $\chi : \mathbf{g} \to \mathbf{h}/\mathcal{W}$, $\chi : G \to T/\mathcal{W}$ and the resolution of the nilpotent, unipotent variety, Brieskorn was able to proof the conjecture, stated by Grothendieck, that the quotient singularities emerge as the intersection of the nilpotent variety and a transversal slice to the orbit of a subregular nilpotent element in the simple Lie algebra $\mathbf{g}$ of type $A$, $D$ or $E$, or analogously in the algebraic group $G$ of this type with unipotent instead of nilpotent.

In view of this theorem, which we will illustrate in the final section of this thesis, it is interesting to see what Klein once said about his work and its relation to the work of Lie:

> Meine Verpflichtungen gegen Hrn.Lie gehen in die Jahre 1879-70 zurück, wo wir in engem Verkehre mit einander unsere Studienzeit in Berlin und Paris abschlossen. Wir fassten damals gemeinsam den Gedanken, überhaupt solche geometrische oder analytische Gebilde in Betracht zu ziehen, welche durch Gruppen von Aenderungen in sich selbst transformirt werden. Dieser Gedanke is für unsere beiderseitigen späteren

> Arbeiten, soweit dieselben auch auseinander zu liegen scheinen, bestimmt geblieben. Während ich selbst in erster Linie Gruppen discreter Operationen in Auge fasste und also insbesondere zur Untersuchung der regulären Körper und ihrer Beziehung zur Gleichungstheorie geführt wurde, hat Hrn.Lie von vorneherein die schwierigere Theorie der continuirlichen Transformationsgruppen und somit die Differentialgleichungen in Angriff genommen.

These two viewpoints turned out to have more connections then Klein could foresee.

P. Slodowy has written various articles and gave lectures about this topic the last twenty years, see [32], [33], [34], [35], [36]. There is a book of Lamotke, see [23], that exhibits the correspondences between the platonic solids, their symmetries and the quotient singularities and resolutions. On the connection of the $ADE$ classification with quivers and the representation theory of algebras, there is a recent survey article by Idun Reiten, see [30]. We do not discuss this connection in this thesis.

There are many recent developments in the subject discussed here. Its widespread presence and significance in mathematics and physics, i.e. string theory, caused many to study it, both mathematicians and physicists. It is of great interest to see if, and if so, in what way the theory can be generalized. We will conclude this thesis by making some remarks about this. However in this thesis my goal was to give a clear picture of the list of connections stated above. On various occasions the large amount of theory that is covered by these different topics has made it necessary to omit details and proofs, many of which are standard. They can be found in the references. I have nevertheless tried to illustrate the discussed theory with examples.

## 2 Platonic Solids and symmetry

The platonic solids are a class of objects that have been present in mathematics since ancient times. They were already described and classified by the Greek mathematicians.

Euclid's work culminates in the description and classification of the regular polyhedra. These results are contained in Book 13 of Euclid's elements; see [12]. Plato (427-347 B.C.) used the regular polyhedra to describe matter in terms of a mathematical model. In his dialogue "Timaios", he associates the four solids tetrahedron, cube, octahedron and icosahedron to the basic four elements of matter: fire, earth, air and water. The dodecahedron represented the universe. Roughly 2000 years later, in 1596, Kepler used the platonic solids in his description of the solar system. He associated a platonic solid to each of the six planets known at that time, in such a way that when inscribed in each other this gave an explanation of the configuration of the planets. For a modern account on the regular polyhedra see [7] or [44].

We recall briefly, that a polyhedron $P$ is an intersection of finitely many closed half-spaces $H_i$ in 3-dimensional Euclidean space $E$. Let $P = \cap_{i=1}^{m} H_i$

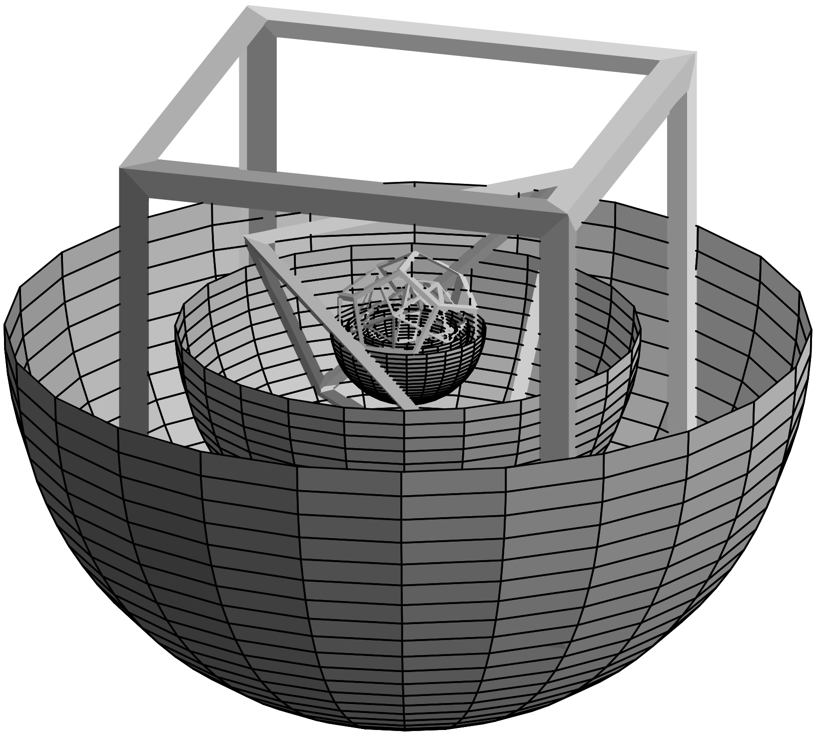


Figure 1: Kepler's proof of the configuration of orbits of the planets

be a bounded polyhedron. $P$ is a convex compact subset of $E$. Let the $V_i$ be the defining planes of the $H_i$. Then a finite intersection of these planes intersected with $P$ gives us, if not empty, a polygon, a line segment, or a point. These are called the faces, edges and vertices of $P$ respectively. Such a P obtained is called a regular polyhedron, or a platonic solid, if its faces are congruent regular polygons, and if the number of these faces meeting in a vertex is the same for all vertices.

For every $q \geq 3$ there exists a unique (up to size) regular convex polygon with $q$ edges and $q$ vertices. It's group of rotation symmetries is the cyclic group of order $q$:

$$\mathbb{Z}/q\mathbb{Z} \hookrightarrow SO(2),$$

and its full group of symmetries (rotations and reflections), is the dihedral group of order $2q$:

$$\mathbb{D}_q = \mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \hookrightarrow O(2).$$

So we can define a regular polyhedron $P$ by a pair of integers $\{q, p\}$, the Schläfli symbol, each face of $P$ being a regular $q$-gon and $p$ of them meeting in each vertex of $P$. A regular $q$-gon has interior angle $\alpha_q = \pi(1 - \frac{2}{q})$, whence

$$2\pi > p\alpha_q = \pi(p - \frac{2p}{q}).$$

This gives us:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{2} > 1. \tag{1}$$

We see that the only possibilities for $\{q, p\}$, if we admit the degenerate cases in which one of them is 2, are the following:

| $\{q,p\}$ | $P$ | $f$ | $e$ | $v$ |
|---|---|---|---|---|
| $\{q,2\}$ | dihedron | 2 | $q$ | $q$ |
| $\{2,p\}$ | hosohedron | $p$ | $p$ | 2 |
| $\{3,3\}$ | tetrahedron | 4 | 6 | 4 |
| $\{4,3\}$ | cube | 6 | 12 | 8 |
| $\{3,4\}$ | octahedron | 8 | 12 | 6 |
| $\{5,3\}$ | dodecahedron | 12 | 30 | 20 |
| $\{3,5\}$ | icosahedron | 20 | 30 | 12 |

Table 1: Schläfli symbols. $f$ faces, $e$ edges and $v$ vertices

The dihedron consists of 2 regular $q$-gons glued together at the edges. The hosohedron can be considered the dual of this; a sphere divided into $q$

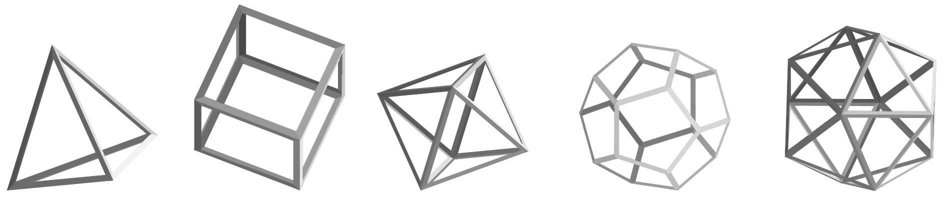



Table 2: The five platonic solids

equal parts by great circles on the sphere intersecting in the north and south pole .

The reason of the interest in these objects is their high degree of regularity which is represented by the rich symmetry groups they possess. We will therefore for each object in the list above determine its group of rotation symmetries $G$.

Clearly the axis of a rotation symmetry should always intersect $P$ in a vertex ,the midpoint of a face, or in the midpoint of an edge. We will denote the order of the respective symmetries with $n_1$, $n_2$, $n_3$. This gives us the following symmetry groups $G$, with orders $N$:

| $P$ | $n_1$ | $n_2$ | $n_3$ | $N$ | $G$ |
|---|---|---|---|---|---|
| dihedron | 2 | $p$ | 2 | $2p$ | $D_p$ |
| hosohedron | $q$ | 2 | 2 | $2q$ | $D_q$ |
| tetrahedron | 3 | 3 | 2 | 12 | $T \cong A_4$ |
| cube | 3 | 4 | 2 | 24 | $O \cong S_4$ |
| octahedron | 4 | 3 | 2 | 24 | $O \cong S_4$ |
| dodecahedron | 3 | 5 | 2 | 60 | $I \cong A_5$ |
| icosahedron | 5 | 3 | 2 | 60 | $I \cong A_5$ |

Table 3: Symmetry groups of the regular polyhedra

Let us mention the well known duality; If we take a regular polyhedron $P$ we get a dual $P'$; it has its vertices in the midpoints of faces of $P$. This gives the duality (dihedron,hosohedron), (cube,octahedron) , (dodecahedron,icosahedron). The tetrahedron is self-dual. The symmetry groups of dual solids are isomorphic.

We get the descriptions as permutation groups by looking at the action of $G$ on a well chosen subset of $P$, namely the set of 4 vertices of the tetrahedron, the set of 4 body-diagonals of the cube, and the set of 5 so called “Dreibeine” in the icosahedron, each consisting of 3 pairwise orthogonal axes through the midpoints of opposite edges.

If we put the center of gravity of the above objects in the origin of $E$, we see that the groups $G$ are naturally embedded in $SO_3(\mathbb{R})$. In fact we will see that up to the case of the cyclic group of order $N$ they exhaust all the possibilities of finite subgroups of $SO_3(\mathbb{R})$.

# 3 Finite subgroups of $SU_2(\mathbb{C})$

Let $SU_2(\mathbb{C}) \subset SL_2(\mathbb{C})$ be the special unitary group, the subgroup of unitary transformations in $GL_2(\mathbb{C})$ with determinant 1. It has a group action on the complex projective line $\mathbb{P} := \mathbb{P}^1(\mathbb{C})$:

$$SU_2(\mathbb{C}) \times \mathbb{P} \longrightarrow \mathbb{P}$$

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (z:1) \right) \longmapsto (\frac{az+b}{cz+d} : 1),$$

for all points $(z:1)$. The point at infinity $(1:0) = \infty$ goes to $(ac^{-1}:1)$ if $c \neq 0$ and else to $\infty$. The element $-I$ acts trivially so the action factors through an action of $PSU_2(\mathbb{C})$.

Take a finite subgroup $\Gamma \subset SU_2(\mathbb{C})$ and let $G$ be its image in $PSU_2(\mathbb{C})$. We want to classify the possible groups $G$. To do so assume that $G$ is not trivial and consider its action on $\mathbb{P}$. Let $F \subset \mathbb{P}$ be the set of points which are fixed by some non-identity element of $G$. Each non-identity element has exactly 2 fixed points, hence $F$ is a finite set and $G$ permutates it. Choose representatives $p_1$,...,$p_d \in \mathbb{P}$ of the distinct orbits of $G$ in $F$. Let $G_i$ be the corresponding stabilizer subgroups of these points in $G$;

$$G_i = G_{p_i} = \{g \in G | g \cdot p_i = p_i\}$$

and let $n$, $n_i$, be the orders of the groups $G$, $G_i$, respectively. We see now that

$$\begin{aligned} 2(n-1) &= \sum_{p \in \mathbb{P}} (|G_p| - 1) = \sum_{p \in F} (|G_p| - 1) \\ &= \sum_{p \in F} (|G_p|) - |F| = \sum_{i=1}^{d} \frac{n}{n_i} n_i - \sum_{i=1}^{d} \frac{n}{n_i} \end{aligned}$$

Rewriting this[1] gives us

$$d + \frac{2}{n} = 2 + \sum_{i=1}^{d} \frac{1}{n_i}. \tag{2}$$

Clearly all $n_i \geq 2$, so our equation implies that we must have $d+\frac{2}{n} \leq 2+\frac{d}{2}$, which gives us $d = 1$, 2 or 3.

$d = 1$ Then $2 = n + \frac{n}{n_i}$, so $n = 1$, $G$ consists of the identity element alone.

$d = 2$ Then $\frac{2}{n} = \frac{1}{n_1} + \frac{1}{n_2}$, whence $n = n_1 = n_2$ and $G = G_1 = G_2$ cyclic groups of order $n$.

$d = 3$ Then the equation becomes

$$1 + \frac{2}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}. \tag{3}$$

so what we see is that $\sum_{i=1}^{3} \frac{1}{n_i} > 1$, so if we take $2 \leq n_3 \leq n_2 \leq n_1$, we see that $n_3 = 2$ and $n_2 \leq 3$. Using this we find the following possibilities for $(n, n_1, n_2, n_3)$:

$$(2p, p, 2, 2), (12, 3, 3, 2), (24, 4, 3, 2), (60, 5, 3, 2). \tag{4}$$

These are the so-called polar configurations of our subgroups of $PSU_2(\mathbb{C})$.

$\mathbb{P}$ is isomorphic to the sphere $S^2 \subset E$: We identify $\mathbb{C} \subset \mathbb{P}$ with the plain in $E$ through the equator and extend the stereographic projection by sending the point at infinity to the north pole of the sphere. This isomorphism gives us an isomorphism of groups:

$$PSU_2(\mathbb{C}) \cong SO_3(\mathbb{R})$$

so there is a bijection of the finite subgroups of these groups. We see that the rotation-symmetry groups of the objects of the last paragraph give realizations of the possible polar configurations, except for the (trivial) cyclic case. A realization of the cyclic group of order $n$ as a rotation symmetry group, is given by the pyramid with $n$ sides. We can therefore conclude:

**Theorem 3.1.** *If $G$ is a finite subgroup in $SO_3(\mathbb{R})$, then it is isomorphic to a cyclic group of order $n$, $C_n$, the dihedral group of order $2n$, $D_n$, the alternating group on 4 elements, $A_4$, the symmetric group of 4 elements, $S_4$, or the alternating group on 5 elements, $A_5$.*

[1] This equality is also known as the theorem of Luroth.

*Proof.* The cyclic case is clear. The rest follows from comparing the results in table 3 and the listed polar configurations in (4). □

We now use the double covering:

$$\pi : SU_2(\mathbb{C}) \longrightarrow SO_3(\mathbb{R}).$$

to get a classification of all possible finite subgroups of $SU_2(\mathbb{C})$. If $G$ is a finite subgroup of $SO_3(\mathbb{R})$ we see that $|\pi^{-1}(G)| = 2|G|$.

**Proposition 3.2.** *Let $\Gamma \subset SU_2(\mathbb{C})$ be a finite subgroup. Then $\Gamma = \pi^{-1}(G)$, for some $G \subset SO_3(\mathbb{R})$ if and only if $\Gamma$ is not cyclic of odd order.*

*Proof.* $\Gamma = \pi^{-1}(G)$ if and only if $-I \in \Gamma$, which is true if and only if $|\Gamma|$ is even. In the case that $|\Gamma|$ is odd we see that $\Gamma \cong \pi(\Gamma)$, which makes $\Gamma$ a subgroup of $SO_3(\mathbb{R})$ of odd order, hence cyclic by the last theorem. □

The subgroups $\Gamma$ that are inverse images of subgroups in $SO_3(\mathbb{R})$ are called binary. There is the binary cyclic group $\mathcal{C}_{2n} = C_n^*$ of order $2n$, the binary dihedral groups $\mathcal{D}_n = D_n^*$, of order $4n$, the binary tetrahedral group $\mathcal{T} = T^*$ of order 24, the binary octahedral group $\mathcal{O} = T^*$ of order 48, and the binary icosahedral group $\mathcal{I} = I^*$ of order 120. And there is the cyclic group $\mathcal{C}_{2n-1}$ of odd order.

We have a classification of the finite subgroups of $SU_2(\mathbb{C})$. Since every finite subgroup of $SL_2(\mathbb{C})$ is conjugated to a subgroup in $SU_2(\mathbb{C})$ we also have a classification of the finite subgroups of $SL_2(\mathbb{C})$.

Let us finally mention here that the three special orbits $Gp_i$ in $PSU_2(\mathbb{C})$are corresponding to the three special sets of points in the regular polyhedra, namely the set of vertices, the set of midpoints of edges, and the set of midpoints of faces.

# 4 Invariant theory of binary polyhedral groups

Let us first collect some general remarks and results about the theory of invariants. Take V a finite dimensional complex vector space, let $(e_i)_{i=1}^n$ be a basis of $V$ and let $f_i \in V^*$ be the corresponding linear functions on $V$ defined by $f_i(e_j) = \delta_{ij}$. Then the $f_i$ generate the symmetric algebra $S = S(V^*)$ of $V$, which is isomorphic to the polynomial algebra on $V$, $\mathbb{C}[Z_1, ..., Z_n]$. $S$ is a graded algebra, its components being the homogeneous polynomials of degree $d$;

$$S = \bigoplus_{d \geq 0} S_d.$$

Let $GL(V)$ denote the group of all invertible linear transformations of $V$. Every subgroup $G$ of $GL(V)$ has an action on the algebra $S$, coming from the contragradient action:

$$g.f(v) = f(g^{-1}.v) \quad , g \in G,\ f \in S,\ v \in V.$$

This action preserves the grading.

We denote the subalgebra of $G$-invariant polynomial functions by $S^G$;

$$S^G := \{f \in S | g.f = f \quad \forall g \in G\} = \bigoplus_{d \geq 0} (S^G \cap S_d).$$

**Theorem 4.1.** *For a reductive linear algebraic group the algebra of invariants $S^G$ is finitely generated over $\mathbb{C}$. In particular for finite $G$ the algebra $S^G$ is finitely generated over $\mathbb{C}$.*

*Proof.* This result in the case of $G$ finite was already proven by E. Noether. For a proof of the more general statement see [38]. □

The invariant theory for finite subgroups of $SU_2(\mathbb{C})$ was solved by Klein in his study concerning quintic equations, see [18]. We will follow mainly the same approach to establish the results he obtained. Other sources are [2], [23], [44] and Springer [38]. Springer uses some invariant theory of complex reflection groups. Finally we mention here the article of Pinkham [28]. He uses more advanced methods from algebraic geometry. We will first state the results and then (partly) give a proof of them.

What Klein found was that the ring of invariants $S^\Gamma$, $\Gamma$ a finite subgroup of $SU_2(\mathbb{C})$, is generated by 3 fundamental invariants $X$, $Y$, $Z$, subject to one essential relation $R(X, Y, Z) = 0$, which is weighted homogeneous;

$$S^\Gamma = \mathbb{C}[Z_1, Z_2]^\Gamma = \mathbb{C}[X, Y, Z]/(R) =: V.$$

We see that the map $\Phi : \mathbb{C}^2 \to \mathbb{C}^3$,

$$\Phi(z_1, z_2) = (X(z_1, z_2), Y(z_1, z_2), Z(z_1, z_2)),$$

factorizes over the quotient space $\mathbb{C}^2/\Gamma$. The image of $\Phi$ is the zero set of $R$, $V(R)$, so $\Phi$ induces an isomorphism of varieties $\mathbb{C}^2/\Gamma \cong V$.

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\Phi} & V(R) \subset \mathbb{C}^3 \\ \downarrow & & \uparrow id \\ \mathbb{C}/\Gamma & \xrightarrow{\sim} & V(R) \end{array}$$

| $\Gamma$ | $R(X,Y,Z)$ | $(d; w_1, w_2, w_3)$ |
|---|---|---|
| $\mathcal{C}_n$ | $X^n + YZ$ | $(2n; 2, n+1, n+1)$ |
| $\mathcal{D}_n$ | $X^{n+1} + XY^2 + Z^2$ | $(2n+2; 2, n, n+1)$ |
| $\mathcal{T}$ | $X^4 + Y^3 + Z^2$ | $(12; 3, 4, 6)$ |
| $\mathcal{O}$ | $X^3Y + Y^3 + Z^2$ | $(18; 4, 6, 9)$ |
| $\mathcal{I}$ | $X^5 + Y^3 + Z^2$ | $(30; 6, 10, 15)$ |

Table 4: Essential relations R for the fundamental invariants of $G$.

For later use we make the following note here. Let the multiplicative group of the complex numbers $\mathbb{C}^*$ act on $l$-dimensional complex space $\mathbb{C}^l$ with integer weights $w_1, ..., w_l \in \mathbb{Z}$; for $x = (x_1, ..., x_l) \in \mathbb{C}^l$, and $t \in \mathbb{C}^*$,

$$t.x = t.(x_1, ..., x_l) = (t^{w_1}x_1, ..., t^{w_l}x_l).$$

Let us recall that each linear $\mathbb{C}^*$-action is of this type. If there is another such action on $p$-dimensional space with weights $d_1, ..., d_p$, we say that a $\mathbb{C}^*$-equivariant map $f : \mathbb{C}^l \to \mathbb{C}^p$ is weighted homogeneous of type $(d_1, ..., d_p; w_1, ..., w_l)$, with weights $w_1, \ldots, w_l$ and degrees $d_1, \ldots, d_p$. The monomials $a_{i_1,...,i_l}x_1^{i_1} \cdots x_l^{i_l}$ that appear in a component $f_i$, a weighted homogeneous polynomial of type $(d_i; w_1, ..., w_l)$, satisfy $\sum_{j=1}^{l} i_j w_j = d_i$.

We have listed the results for the fundamental invariants $X$, $Y$ and $Z$, conveniently chosen, in table 4 and we have added the weighted homogeneous type of $R$.

**Proposition 4.2.** *Let $f : \mathbb{C}^3 \to \mathbb{C}$ be weighted homogeneous of one of the types in table 4 and $R$ the corresponding polynomial. Then there is a $\mathbb{C}^*$-equivariant automorphism $\alpha$ of $\mathbb{C}^3$ such that $f \circ \alpha = R$.*

*Proof.* For a proof of this statement see [32]. □

Now let us see how to obtain these results. To start with we have a finite subgroup $\Gamma$ of $SU_2(\mathbb{C})$, we take $V = \mathbb{C}^2$ and we want to determine invariant polynomials. We know that the action preserves the grading of $\mathbb{C}[Z_1, Z_2]$, so we can restrict our inquiry to the analysis of the action on homogeneous polynomials.

The cyclic and dihedral case are easy and can be dealt with directly. We compute invariants for the generators of $\Gamma$. They give us the invariants for $\Gamma$.

Let $\Gamma$ be cyclic of order $N$. A generator for $\Gamma$ will be $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, with $\zeta$ a primitive $N$-th root of unity; it acts on $\mathbb{C}[Z_1, Z_2]$ by $Z_1 \mapsto \zeta^{-1}Z_1$, $Z_2 \mapsto \zeta Z_2$.

We get the invariants $X = Z_1Z_2$, $Y = Z_1^N$ and $Z = Z_2^N$, which span the whole of $S^\Gamma$ and satisfy the relation $X^N = YZ$.

Let $\Gamma$ now be binary dihedral of order $4N$. Generators of $\Gamma$ are $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. The first has invariants listed above, the second has the invariants $Z_1^2 - Z_2^2$, $Z_1Z_2(Z_1^2 + Z_2^2)$ and $Z_1^2Z_2^2$. We get invariants of $\Gamma$, $X = Z_1^2Z_2^2$, $Y = Z_1^{2N} + (-1)^N Z_2^{2N}$, $Z = Z_1Z_2(Z_1^{2N} - (-1)^N Z_2^{2N})$. They span the whole of $S^\Gamma$ and satisfy the relation

$$Z^2 - XY^2 = 4(-X)^{N+1}.$$

So if we substitute $-X$ for $X$, $2Y$ for $Y$, and $2Z$ for $Z$, they satisfy the relation $R(X, Y, Z) = 0$ listed above.

In the other cases this method is not a good choice. The generators of $\Gamma$ get more complicated and the invariants of them have high degrees. It is easier to calculate the semi-invariants of $\Gamma$.

**Definition 4.3.** *$f$ is a semi-invariant of $\Gamma$ if*

$$\exists \chi \in HOM(\Gamma, \mathbb{C}^*) \textit{ such that } \quad \gamma.f = \chi(\gamma)f \quad \forall \gamma \in \Gamma.$$

Such characters $\chi$ of $\Gamma$ factor through the abelianized group, $\mathcal{A}\Gamma$.

We have an action of $\Gamma$ on $\mathbb{P}$ which factors through the action of $G = \pi(\Gamma) \subset SO_3(\mathbb{R})$ and we take as before representatives $p_1$,..,$p_d$ of the distinct orbits of the action of $G$ on $F$, the set of points fixed by some non-identity element in $G$. So we can define an action on the Abelian group of divisors on $\mathbb{P}$, $\mathcal{D}$:

$$\gamma.D(x) = D(\gamma^{-1}.x) \text{ , with } D \in \mathcal{D} \text{ and } x \in \mathbb{P}.$$

Every homogeneous polynomial $f$ of degree $d$ defines a divisor $D_f \in \mathcal{D}$ of degree $d$. Namely for $x \in \mathbb{P}$ we let $D(x)$ be the order of vanishing of $f$ on the line in $\mathbb{C}^2$ defined by $x$, and a divisor likewise defines a homogeneous polynomial, uniquely up to a scalar factor.

If $D_f$ is $\Gamma$-invariant we see that $f$ is $\Gamma$-semi-invariant. The $\Gamma$-invariant divisors clearly form a subgroup of $\mathcal{D}$. An orbit $\Sigma \subset \mathbb{P}$ defines a $\Gamma$-invariant divisor $D$;

$$D(x) = 1 \text{ if } x \in \Sigma \quad , D(x) = 0 \text{ else.}$$

Such $D$ are called simple and are denoted by the same letter $\Sigma$. In our case we have 2 (cyclic case) or 3 exceptional orbits $\Sigma_i = \Gamma p_i$. They define exceptional divisors $\Sigma_i$, and the following semi-invariants $f_i$, of degree $|Gp_i|$:

$$f_i(Z_1, Z_2) = \prod_{(a_1:a_2) \in G(p_i)} (a_2Z_1 - a_1Z_2)$$

With every divisor $D$ we get a well-defined character $\chi_D$, since we can associate a semi-invariant $f$ with $D$.

**Proposition 4.4.** *Let $\Gamma$ be a finite subgroup of $SU_2(\mathbb{C})$ with its canonical action on polynomials in two variables. Let $\mathcal{D}$ be the abelian group of divisors on $\mathbb{P}$, with an action of $\Gamma$ defined as above. Then the following holds:*

1. *Every positive invariant divisor $D \in \mathcal{D}$ can uniquely (up to order) be written as a finite sum:*
$$D = \sum_{i=1}^{k} r_i \Sigma_i$$
*of different simple divisors $\Sigma_i$, with $r_i \in \mathbb{Z}_{>0}$.*

2. *There is a bijection between the group of positive $\Gamma$-invariant divisors and the ring of $\Gamma$-semi-invariant homogeneous polynomials; Take $D \in \mathcal{D}$ positive and invariant. It defines a homogeneous semi-invariant $f$ by*
$$f(Z_1, Z_2) = \prod_{a=(a_1:a_2)\in\mathbb{P}} (a_2 Z_1 - a_1 Z_2)^{D(a)}.$$
*Take $f$ a semi-invariant. It defines a divisor $D \in \mathcal{D}$ by*
$$D([a_1 : a_2]) = ord_f((a_1, a_2)).$$

3. *If $f$ is a $\Gamma$-semi-invariant homogeneous polynomial, then $f$ is $[\Gamma, \Gamma]$-invariant.*

*Proof.* Take a positive non zero divisor $D$. $D \neq 0$ in a finite number of points. $D$ is constant on orbits and there are only finitely many orbits, say $\Sigma_1, ..., \Sigma_k$ on which $D$ is strictly positive, of value $r_1, ..., r_k$. This proofs the first part of the proposition. The second part follows from remarks made above. For the third part, let $f$ be $\Gamma$-semi-invariant. The character $\chi_f$ factorizes through $\mathcal{A}\Gamma = \Gamma/[\Gamma, \Gamma]$. □

In the icosahedral case $\mathcal{A}\Gamma$ is trivial and we get the 3 fundamental invariants directly from the three exceptional orbits, of order 12, 20, and 30 respectively. It can be shown that the the space of invariant homogeneous polynomials of degree 60 is 2-dimensional. This gives us that these three invariants span the whole of $S^\Gamma$ and by scaling them they satisfy the relation in the list.

In the other 2 cases, the tetrahedral and the octahedral case, we get three semi-invariants, coming from the exceptional orbits. Here $\mathcal{A}\Gamma$ is not trivial so we need some explicit calculation of the characters to get the results listed above. See the references for details.

# 5 Kleinian singularities

## 5.1 Characterizations of Kleinian singularities

We have shown that the quotient variety $\mathbb{C}/\Gamma$ is isomorphic to the affine orbit variety $V$. We see that in all cases we have a unique isolated singularity at 0. These particular singularities are called Kleinian singularities, but they are also denoted by rational double points, quotient singularities or simple singularities. They form a special class of singularities, having various equivalent characterizations.

**Proposition 5.1.** *The following characterizations of singular germ $(V, 0)$ of a surface are equivalent:*

1. *Let the finite group $G \subset SL_2(\mathbb{C})$ act naturally on complex 2-dimensional space. Then the orbit space $\mathbb{C}^2/G$ is a complex space and $(V, 0)$ is isomorphic to the germ of the set $(\mathbb{C}^2/G, 0)$. Germs of this type are called quotient singularities.*

2. *$(V, 0)$ is isomorphic to the germ $(V', 0)$ of the affine variety $V' = \mathbb{C}[X, Y, Z]/ < R >$, with $R$ one of the polynomials from table 4.*

3. *Let $\rho : W \to V$ be a minimal resolution of the singularity $(V, 0)$. Then the exceptional set $\rho^{-1}(0)$ is a finite union of projective lines, which have intersection matrix the negative of a Cartan matrix of a Dynkin diagram of type $A$, $D$ or $E$.*

4. *The singularity $(V, 0)$ is normal, has embedding dimension $\leq 3$ and is rational: Let $\rho : V^* \to V$ be a minimal resolution of $0 \in V$. Then the higher direct images of the structure sheaf are zero;*

$$(R^i \rho_* \mathcal{O}_{V^*})_0 = 0 \ , \forall i > 0,$$

*Proof.* 1 $\iff$ 2, follows from section 4. To proof the equivalence 1 $\Leftrightarrow$ 3, we will give resolutions of the quotient singularities and show that they are of the type stated here. Moreover they will turn out to be characterizing; the minimal resolution is unique. For the equivalence with 4 and other characterizations of Kleinian singularities we refer to [8]. □

A point $p$ on an analytic variety is called regular if a neighborhood of $p$ is isomorphic to a complex manifold. A non-regular point is called singular. For a general account of singularities see [2]. Furthermore the survey article of Brieskorn [5] is recommended. He takes the hyper-geometric differential equation as a starting point. It is closely related to our discussion and has a long and rich history in mathematics.

## 5.2 Resolutions of Kleinian singularities

The goal of this section is to give a minimal resolution of the quotient singularity, which as stated gives a characterization of the singularity. Let us first give some definitions.

**Definition 5.2.** *A modification of a variety $V$ at a point $v$ is a morphism*

$$\mu : \overline{V} \longrightarrow V$$

*with the following properties:*

1. *$\mu$ is proper (In the case of a complex variety this means that the inverse image of compact subsets are compact).*
2. *$\mu$ restricted to $\mu^{-1}(V\backslash\{v\})$ is an isomorphism to its image $V\backslash\{v\}$ and $\mu^{-1}(V\backslash\{v\})$ is dense in $\overline{V}$.*

The subvariety $E := \mu^{-1}(v)$ is called the exceptional set.

**Definition 5.3.** *And if $(V, v)$ is a singularity it is called a resolution of the singularity if in addition*

3. *every point of $\overline{V}$ is regular.*

For curve and surface singularities it is since long known that resolutions exist. In characteristic zero there is a theorem by Hironaka that states existence of resolutions of singularities in the general case. For the theory of surface singularities see Laufer [24]. The method used to obtain a resolution is the blow-up of singular points. The singular point is replaced by all the lines through it. A resolution $\rho$ is minimal if all other resolutions factor through $\rho$. Minimal resolutions exist and are unique up to isomorphism.

We now give a construction for the minimal resolutions of the quotient singularities coming from the binary polyhedral groups. We will follow mainly the approach of Lamotke, see [23].

As usual the cyclic case is the easiest one. Let $\Gamma \subset SU_2(\mathbb{C})$ be the cyclic subgroup of order $N$ in $SU_2(\mathbb{C})$,

$$\Gamma =< \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} > .$$

We will give a resolution of the corresponding quotient singularity $\mathbb{C}^2/\Gamma$. The singularity is isomorphic to the hyperplane singularity $(V, 0)$ defined by the relation $X^N = YZ$.

We consider the line bundle $\Sigma(b)$ (This is $\mathcal{O}(-b)$),

$$\Sigma(b) := \{((z_1, z_2), (w_1 : w_2)) \in \mathbb{C}^2 \times \mathbb{P} | z_1 w_2^b = z_2 w_1^b\},$$

and we give holomorphic charts $U_1$ and $U_2$ for $\Sigma(b)$, induced from the 2 charts covering $\mathbb{P}$:

$$U_i := \{(z, w) \in \Sigma(b) | w_i \neq 0\} \ ,$$

$$\Sigma(b) \supset U_1 \to \mathbb{C}^2 \ , \ (z, w) \mapsto (z_1, \frac{w_2}{w_1}) \ , \ w_1 \neq 0$$

and

$$\Sigma(b) \supset U_2 \to \mathbb{C}^2 \ , \ (z, w) \mapsto (z_2, \frac{w_1}{w_2}) \ , \ w_2 \neq 0.$$

We see that the $U_i$ are isomorphic to $\mathbb{C}^2$ and the transition is given by $\phi : (u, v) \mapsto (\frac{1}{v}, u^b v)$. The $z_1$ and $z_2$ are called linear fiber coordinates and $\frac{w_1}{w_2}$, $\frac{w_1}{w_2}$ affine base coordinates.

We use this to form a manifold $M$ by pasting more of these copies together. We take

$$M = \bigcup_{i=1}^{N} U_i \ , \text{ with } U_i \cong \mathbb{C}^2 \ , \text{ and } U_i \cup U_{i+1} =: L_i \cong \Sigma(2) \cong T^*\mathbb{P},$$

the $U_i$ having coordinates $(u_i, v_i)$ such that $u_i$ is an affine base coordinate of $L_{i-1}$ and a linear fiber coordinate of $L_i$ and $v_i$ is a linear fiber coordinate of $L_{i-1}$ and an affine base coordinate of $L_i$.

The transition, pasting of the $U_i$, is done by

$$\begin{array}{rcl} \phi_i : U_i \backslash \{v_i = 0\} & \longrightarrow & U_{i+1} \backslash \{u_{i+1} = 0\} \\ (u_i, v_i) & \mapsto & (\frac{1}{v_i}, u_i v_i^2) = (u_{i+1}, v_{i+1}) \end{array}$$

for $i = 1, ..., N-1$.

$$M = \bigcup_{i=1}^{N-1} L_i \ , \ L_i \cap L_{i+1} = U_{i+1}$$

We have the following result:

**Proposition 5.4.** *The minimal resolution for $\mathbb{C}^2/\Gamma$, $\Gamma$ cyclic of order $N$, is given by*

$$\begin{array}{rl} \rho : M \to & V \\ (u_i, v_i) \mapsto & (x, y, z) = (u_i v_i, u_i^i v_i^{i-1}, u_i^{N-i} v_i^{N-i+1}). \end{array}$$

*The exceptional set of $\rho$, $E$, is the union of the zero sections of the line bundles $L_i$,*

$$E = \bigcup_{i=1}^{N-1} E_i \ , \textit{ with } E_i := \{u_i = v_{i+1} = 0\}$$

*Here $E_i$ and $E_{i+1}$ intersect transversely in one point and $E_i \cap E_j = \emptyset$ for $|i-j| > 1$. The self-intersection number of the components $E_i$ is $-2$.*

*Proof.* We will only give a sketch. For details see [23]. □

The key ingredient of the proof is the following result about cyclic quotients:

Let $H$ be a finite cyclic subgroup in $GL_n(\mathbb{C})$. Up to linear coordinate transformations we can assume that $H$ is generated by $diag(\zeta_r, \zeta_s)$, with $\zeta_i$ a primitive i-th root of unity. Take $p \in \mathbb{Z}$ such that $rp$ is the least common multiple of $r$ and $s$ and let $H^*$ be the cyclic subgroup of order $r$ generated by $diag(\zeta_r, \zeta_s^p)$.

**Lemma 5.5.** *The mapping $f(z_1, z_2) = (z_1, z_2^p)$ induces an isomorphism of complex spaces*

$$\mathbb{C}^2/H \cong \mathbb{C}^2/H^*.$$

*Proof.* The map $\tilde{f}$ is induced by the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{f} & \mathbb{C}^2 \\ \downarrow \pi & & \downarrow \pi^* \\ \mathbb{C}^2/H & \xrightarrow{\tilde{f}} & \mathbb{C}^2/H^* \end{array}$$

The non-trivial part of the proof is the injectivity of $\tilde{f}$. If $\tilde{f}([u_1, u_2]) = \tilde{f}([v_1, v_2])$ then $\pi^* \circ f(u_1, u_2) = \pi^* \circ f(v_1, v_2)$. This gives us $u_1 = \zeta_r^q v_1$ and $u_2^p = \zeta_s^{pq} v_2^p$, for some integer $q$. We get $u_2 = \zeta_p \zeta_s^q v_2$, and since $\zeta_s^r$ is a primitive $p$-th root of unity there is an integer $l$, such that $\zeta_p = \zeta_s^{rl}$. Take $t = rl + q$. Then clearly $u_1 = \zeta_r^t v_1$ and $u_2 = \zeta_s^t v_2$, whence $[u_1, u_2] = [v_1, v_2]$. □

Using the lemma, we see that $\mathbb{C}^2/\Gamma \cong \mathbb{C}^2/\Gamma^*$, with

$$\Gamma^* = < \begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_{N-1}\zeta_N \end{pmatrix} > .$$

Then we blow-up of the unique singular point $0 \in \mathbb{C}^2/\Gamma^*$ by the canonical map $\Sigma(2) \to \mathbb{C}^2$ and extending the action of $\Gamma^*$ to an action of $\Sigma(2)$, making the following diagram commutative:

$$\begin{array}{ccc} \Sigma(2) & \longrightarrow & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \Sigma(2)/\Gamma^* & \longrightarrow & \mathbb{C}^2/\Gamma^* \end{array}$$

We look now for singularities in $\Sigma(2)/\Gamma^*$. The local charts $U_i$ are $\Gamma^*$-invariant and any singularity must be in the exceptional set which is given by the zero section of $\Sigma(2)$, $\{(z,w) \in \Sigma(2) | z = 0\}$. This is a projective line.

The generator of $\Gamma^*$ acts on $\Sigma(2)$;

$$((z_1, z_2), (w_1 : w_2)) \mapsto ((\zeta_N z_1, \zeta_{N-1}\zeta_N z_2)(w_1 : \zeta_{N-1} w_2)).$$

Now we look in the charts $U_i$. On $U_i$ we can fix $w_i = 1$ and we see that we have the following induced actions:

$$\begin{array}{ll} \text{on } U_1: & (z_1, w_2) \mapsto (\zeta_N z_1, \zeta_{N-1} w_2) \\ \text{on } U_2: & (z_2, w_1) \mapsto (\zeta_{N-1}\zeta_N z_2, \zeta_{N-1}^{-1}). \end{array}$$

We get that

$$U_1/\Gamma^* \cong U_1/ < \begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_{N-1} \end{pmatrix} >,$$

but the last is isomorphic to $U_1/\{Id\}$ using our lemma twice. We get that

$$U_2/\Gamma^* \cong U_2/ < \begin{pmatrix} \zeta_{N-1} & 0 \\ 0 & \zeta_{N-1}^{-1} \end{pmatrix} > .$$

We have a cyclic quotient singularity of order one less in one of the charts, whereas the other chart has no singular points.

So we can repeat this construction leading to cyclic quotient singularities of smaller order and we get the construction of our manifold $M$, with all the properties mentioned above.

Minimality of a resolution can be characterized by the fact that it doesn't have exceptional curves with self-intersection $-1$. Hence this follows from the fact that the self intersection number for the components of $E$ is $-2$. In fact the Euler number of $\mathcal{O}(-2)$ is $-2$. This follows from the fact that it admits a meromorphic section $s$ without zeros and with 2 simple poles:

$$s : \mathbb{P} \to \Sigma(2) \ , \ [w_1 : w_2] \mapsto (\frac{w_1^2}{w_1^2 - w_2^2}, \frac{w_2^2}{w_1^2 - w_2^2}, [w_1 : w_2]).$$

**Definition 5.6.** *Let $\rho : M \to V$ be a resolution of the singularity $(V, 0)$, with exceptional set $E$ a union of projective lines; $E = \bigcup_{i \in I} E_i$, $E_i \cong \mathbb{P}$, where the intersections between the components, if not empty, are transversal.*

*Then the graph $\Delta$ of the resolution has a vertex for each component $E_i$ and a edge between $E_i$ and $E_j$ if and only if $E_i \cap E_j \neq \emptyset$ for $i, j \in I$, $i \neq j$.*

We have demonstrated that the quotient singularity for the cyclic group of order $N$ has the Dynkin graph of type $A_{N-1}$ as a resolution graph.

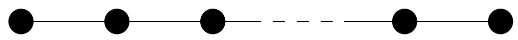


Figure 2: Dynkin diagram $A_{N-1}$

Even in the easiest case possible, namely in the case of $\Gamma$ cyclic of order 2, the real picture of the minimal resolution, see figure 3, gives a good feeling for what is going on. This singularity is defined by the equation $X^2+Y^2-Z^2=0$.

Having dealt with the cyclic case we can proceed with the other. We have a quotient singularity $\mathbb{C}^2/\Gamma$, $\Gamma$ a non-cyclic binary polyhedral group.

We first blow-up the point $0 \in \mathbb{C}^2$,

$$\begin{aligned} \mu : \Sigma(1) &\to \mathbb{C}^2 \\ (z,w) &\mapsto z \end{aligned}$$

and lift the action of $\Gamma$ to $\Sigma(1)$, obtaining a commutative diagram

$$\begin{array}{ccc} \Sigma(1) & \xrightarrow{\mu} & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \Sigma(1)/\Gamma & \xrightarrow{\mu^*} & \mathbb{C}^2/\Gamma \end{array}$$

The map $\mu^*$ is a modification of the singularity. We will find a resolution for $\Sigma(1)/\Gamma$, thus resolving $\mathbb{C}^2/\Gamma$.

We look at the action of $\Gamma$ on $\Sigma(1)$, given by $\gamma(z,w) = (\gamma.z, \gamma.w)$. The exceptional set $E = \mu^{-1}(0)$ is isomorphic to the projective line (hence isomorphic to the sphere) and contains all the points which are a fix-point of some non-identity operation in $\Gamma$.

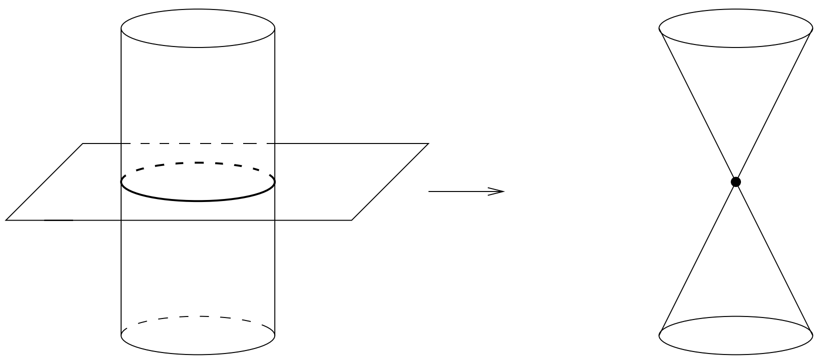


Figure 3: Resolution of cyclic quotient singularity of order 2

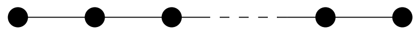
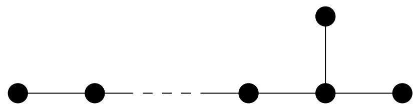
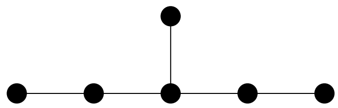
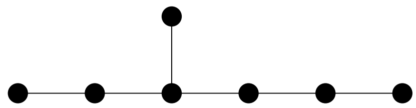
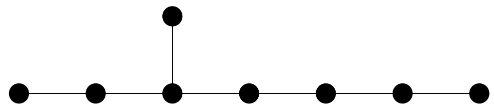
| $\Gamma$ | type of $\Delta$ | $\Delta$ |
|---|---|---|
| $\mathcal{C}_N$ | $A_{N-1}$ |  |
| $\mathcal{D}_N$ | $D_{N+2}$ |  |
| $\mathcal{T}$ | $E_6$ |  |
| $\mathcal{O}$ | $E_7$ |  |
| $\mathcal{I}$ | $E_8$ |  |

Table 5: Resolution graphs for the quotient singularities.

It is easy to see that the action of $\Gamma$ on $E$ factors through the action of $G$ on the sphere, where $G$ is the image of $\Gamma$ in $SO_3(\mathbb{R})$. (Note that $\pm I$ acts trivially.) The projection $\Sigma(1) \to \Sigma(2)$, $(z, w) \mapsto (z^2, w)$ induces an action of $\Gamma$ on $\Sigma(2)$ and we obtain an isomorphism $\Sigma(2)/G \cong \Sigma(1)/\Gamma$. We have obtained the following result:

**Proposition 5.7.** *There is a modification*

$$\psi : \Sigma(2)/G \longrightarrow \mathbb{C}^2/\Gamma.$$

The only singular points come from the exceptional orbits of $\Gamma$ on $E$. We know there are 3 such orbits corresponding to stabilizer subgroups in $G$ which are cyclic subgroups of order $p$, $q$ and 2, the polar configurations of $G$ being (p,q,2). Hence $\Sigma(2)/G$ is a smooth complex surface, except from 3 isolated cyclic quotient singularities of order $p$, $q$ and 2, which all three lie on the exceptional set of $\psi^{-1}(\overline{0}) \cong \mathbb{P}$. We know how to solve these singularities. If we give a resolution for these three singularities we obtain a resolution of the quotient singularity $\mathbb{C}^2/\Gamma$. To proof minimality the criterion of the self intersection numbers is used.

The resolution graph has one special node from where three branches of length $p-1$, $q-1$ and $r-1$ start. This gives the resolution graphs as listed in table 5.

We have sketched the real picture for the resolution of $\mathbb{C}^2/\mathcal{D}_2$ in figure 4.

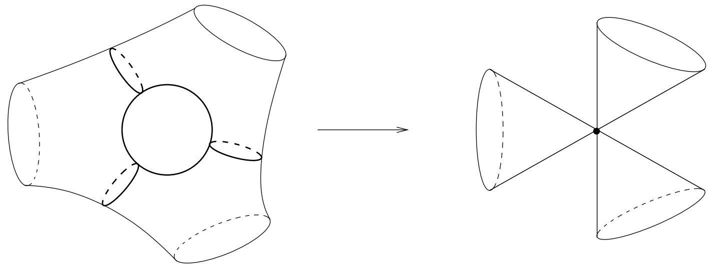


Figure 4: Minimal resolution for $\mathbb{C}^2/\mathcal{D}_2$

## 5.3 Semi-universal deformations

For later use we briefly discuss the notion of deformations. For details see [17] and [27].

**Definition 5.8.** *Let $V$ be a variety. A deformation of $V$ is a pair $(p, i)$, where $p$ is a flat morphism of varieties, $p : X \to (U, u)$ from the total space $X$ to a pointed base space $(U, u)$, and $i : V \to p^{-1}(u)$ is an isomorphism.*

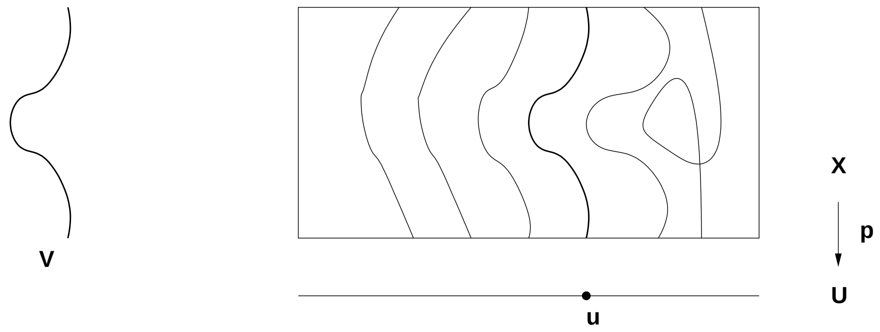



Figure 5: A deformation of a variety

If we have a deformation $(p, i)$, and a morphism of pointed varieties, $\phi : (U', u') \to (U, u)$, we get an induced deformation (by the base change $\phi$), $(\phi * (p), \phi * (i))$, with total space $X \times_U U'$ and base space $U'$.

Two deformations are isomorphic if they have the same pointed base space and there is an isomorphism of the total spaces that is an isomorphism of the special fiber.

The theory of deformations comes from the study of moduli spaces. In our situation we will consider Kleinian singularities, singular germs $(V, 0)$ and we want to find deformations of them. As in the case of resolutions we would also like to have some 'minimal' deformation, if possible unique.

**Definition 5.9.** *A deformation $(p,i)$ of a variety $V$, with total space $X$ and base space $U$, is called semi-universal if and only if both the following hold:*

1. *Up to isomorphism any deformation of $V$ can be induced from $(p,i)$ by some base change $\phi : (U',u') \to (U,u)$.*
2. *The differential $D_{u'}\phi : T_{u'}U' \to T_uU$ is uniquely determined.*

In the present case it turns out to be too much to ask for the base change to be uniquely determined in the special point. The condition of the differential map of the tangent space in the special point to be uniquely determined guaranties that up to isomorphism the semi-universal deformation is unique. See the references for details.

Let $(V,0)$ now be the Kleinian singularity defined as $\mathbb{C}[X,Y,Z]/R$, with $R$ from table 4. Then

$$\mathbb{C}[[Z_1,Z_2,Z_3]]/(R,\frac{\partial R}{\partial Z_1},\frac{\partial R}{\partial Z_2},\frac{\partial R}{\partial Z_3})$$

is finite dimensional so we can choose a basis for it, say $b_1,...,,b_r$.

Define the spaces $X := \{(z,u) \in \mathbb{C}^3 \times \mathbb{C}^r | R(z) + \sum_{i=1}^r u_i b_i(z) = 0\}$ and $(U,u) := (\mathbb{C}^r,0)$. Let the morphisms $p : X \to (U,u)$ and $i : V \hookrightarrow X$ be the projection on the second coordinate, and the embedding of $(V,0)$ into $X$ respectively. Then we have the following result.

**Theorem 5.10.** *The pair $(p,i)$ is a semi-universal deformation of the singularity (V,0).*

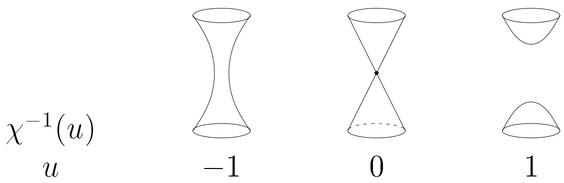
Consider the case that we have a cyclic quotient singularity of order $n$. Then we have the variety $V$ defined by $R = X^n - Y^2 - Z^2 = 0$ hence the quotient algebra

$$\mathbb{C}[X,Y,Z]/(R,\frac{\partial R}{\partial X},\frac{\partial R}{\partial Y},\frac{\partial R}{\partial Z}) = \mathbb{C}[X,Y,Z]/(X^{n-1},Y,Z) = \mathbb{C}[X]/(X^{n-1}).$$

A semi-universal deformation is then given by the second projection from $X$ to $(U,u) = (\mathbb{C}^{n-1},0)$, where

$$X := \{(x,y,z,u_1,...,u_{n-1}) \in \mathbb{C}^{3+n-1} | x^n + \sum_{i=2}^{n} u_i x^{n-i} - y^2 - z^2 = 0\}.$$

n=2 X is a family of surfaces depending on one parameter $u \in \mathbb{C}$. The only singular fiber is the fiber above 0. Let us sketch the possible fibers




n=4 The singularity is of type $A_3$ and we obtain a 3-parameter family of surfaces:

$$x^4 - y^2 - z^2 + u_1 x + u_2 x^2 + u_3 x^3$$

The singular fibers occur exactly for values of $u_1, u_2, u_3$ for which $x^4 + u_1 x + u_2 x^2 + u_3 x^3$ has multiple roots. This defines the discriminant locus, which in this case is a so-called swallow-tail. The singularities of type $A_1$ $A_2$ and $A_3$ also occur as fibers of $\chi$. This phenomenon makes it possible to define a partial ordering on the singularity types, which corresponds with the partial ordering on diagrams. See for instance [34] for details. For the present example we have sketched the singular locus with the occurring singularity types in figure 6.

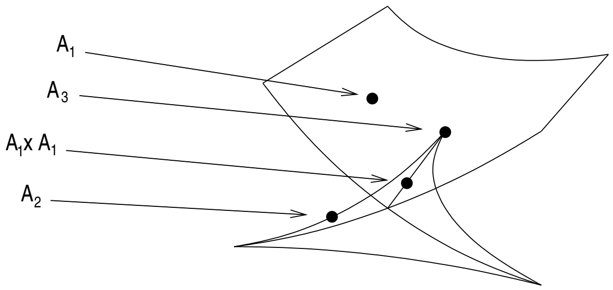



Figure 6: The swallow-tail, parameterizing the singular fibers

# 6 The McKay correspondence

In the last section we have shown that to every finite $\Gamma \subset SU_2(\mathbb{C})$ we can associate a Dynkin diagram, namely the resolution graph of the quotient singularity. Quite recently a direct group theoretical proof of this correspondence has been found by McKay.

Let us consider the finite subgroups of $SU_2(\mathbb{C})$ and their representations. We take such a subgroup $\Gamma$ and we let $\widehat{\Gamma} = \{\rho_0, ..., \rho_r\}$ be the set of equivalence classes of irreducible finite dimensional complex representations of $\Gamma$, with $\rho_0$ the trivial representation.

Now let $\rho : \Gamma \to SU_2(\mathbb{C})$ be the canonical 2-dimensional representation of $\Gamma$. Then we can define for each group $\Gamma$ a matrix $A(\Gamma)$, by decomposing the tensor products:

$$\rho_i \otimes \rho = \sum_{i=0}^{r} A(\Gamma)_{ij}\rho_j.$$

We see that A is a $(l+1) \times (l+1)$-matrix with all $a_{ij} \in \mathbb{Z}_{\geq 0}$.

It was McKay who made the following remarkable observation, see [25], [26]:

**Theorem 6.1.** *With each $\Gamma$ we can associate a complex simple simply laced Lie-algebra $\mathbf{g}$ of rank $r$ such that*

$$A(\Gamma) = 2I - C(\widetilde{\mathbf{g}}) \ ,$$

*where $\widetilde{\mathbf{g}}$ is the affine Kac-Moody Lie-algebra associated to $\mathbf{g}$ and $C(\widetilde{\mathbf{g}})$ its Cartan matrix. The Dynkin diagram that is associated to $\Gamma$ in this way is the extended Dynkin diagram of the diagram associated to $\Gamma$ by the resolution of its quotient singularity.*

*Proof.* This theorem can be proved (observed) by checking, since on both sides of the correspondence the structure is well known. There is a unified proof, not depending on this, by Steinberg, see [43]. $\square$

Let us first collect some results to put this result in perspective. We take a complex simple Lie algebra $\mathbf{g}$ of rank $r$, and in it we take a Cartan sub-algebra $\mathbf{h}$. We obtain a root-system in the dual $\mathbf{h}'$ of $\mathbf{h}$ and in it we make a choice for a positive root-system $\Phi$. We get a set of simple roots and put an ordering on it;

$$\Delta = \{\alpha_1, ..., \alpha_r\} \subset \mathbf{h}'$$

Then the Cartan-matrix of $\mathbf{g}$ is defined by:

$$(C(\mathbf{g}))_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)},$$

the symmetric non-degenerate bilinear form ( , ) on $\mathbf{h}'$ induced by the Killing form of $\mathbf{g}$.

With $\mathbf{g}$ we can associate the affine Kac-Moody Lie algebra mentioned above, which we take modulo its central extension. For details see [16]. The Cartan sub-algebra $\widetilde{\mathbf{h}}$ of $\widetilde{\mathbf{g}}$ has dimension $r+1$ and we can take an ordered set of simple roots of $\widetilde{\mathbf{h}}'$ such that $\widetilde{\Delta} = \{\alpha_0, ..., \alpha_r\}$. This is a basis for $\widetilde{\mathbf{h}}'$ and we can extend ( , ) to a symmetric, but degenerate bilinear form on $\widetilde{\mathbf{h}}'$ by putting:

$$(\alpha_0, \alpha_i) = (-\psi, \alpha_i) \ , \text{ and } (\alpha_0, \alpha_0) = (\psi, \psi) \ ,$$

where $\psi$ is the unique highest root in $\Phi$.

This bilinear form defines the Cartan matrix, of $\widetilde{\mathbf{g}}$, $C(\widetilde{\mathbf{g}})$, which contains the Cartan matrix of $\mathbf{g}$ as a principal $r \times r$ minor. We have the affine Weyl group $\widetilde{\mathcal{W}}$ and to the Cartan matrix we can associate the extended Dynkin diagram; to each simple root we associate a node and there is an edge between two nodes if $C(\widetilde{\mathbf{g}})_{ij} = -1$, this being the case when not equal to zero. The theorem above states that the matrix $A(\Gamma)$, that is obtained for every finite subgroup $\Gamma \subset SU_2(\mathbb{C})$, minus twice the identity matrix is negative the Cartan matrix of $\widetilde{\mathbf{g}}$. We have listed the extended Dynkin diagrams in table 6.

By the correspondence each node of the extended Dynkin diagram corresponds to a unique simple root in $\Delta$ and a representation in $\widehat{\Gamma}$. There is a bijection between the set of irreducible representations and the set of simple roots, $\rho_i \mapsto \alpha_i$, $i = 0, ..., r$. The trivial representation corresponds to the extra node of the extended Dynkin diagram.

Let us take a look at one other way of viewing the correspondence. With each representation $\rho_i$, we get a character $\chi_i$, and we can form the character table. Since the number of irreducible representations of a finite group $\Gamma$ equals the number of conjugacy classes in it, we can choose representatives $x_0 = 1, x_1, ..., x_r$ for the latter and we then form a $(r+1) \times (r+1)$-matrix, the character table, by in the i-th row an j-th column evaluating $\chi_i$ in $x_j$. We now have the following result:

**Proposition 6.2.** *The i-th column of the character table of* $\Gamma$, $\mathbf{v} := (\chi_j(x_i))_j$, *is an eigenvector of the Cartan matrix of* $\mathbf{g}$ *with eigenvalue* $2 - \chi(x_i)$.

*Proof.* Take $x \in \Gamma$. We have $\chi(x)\chi_i(x) = \sum_j A(\Gamma)_{ij}\chi_j(x)$. This gives us $(2-\chi(x_i))\mathbf{v} = 2\mathbf{v} - \sum_{k,j} A(\Gamma)_{kj}\chi_j(x_i) = (2 - A(\Gamma))\mathbf{v} = C\mathbf{v}$. ☐

So we can translate the theorem in terms of the diagrams; if we tensor a representation $\rho_i$ with our fixed 2-dimensional representation $\rho$ we see that this decomposes into the representations $\rho_j$ neighboring $\rho_i$ in the diagram.

**Corollary 6.3.**

$$\rho_j \otimes \rho = \sum_{k \in N(j)} \rho_k \quad , \tag{5}$$

*where* $N(j)$ *is the set of nodes in the diagram that has an edge with the* $j$*-th node.*

We can decompose the vector $\alpha_0 + \psi \in \widetilde{\mathbf{h}}$, which is orthogonal to every other vector in $\widetilde{\mathbf{h}}$;

$$\alpha_0 + \psi = \sum_{i=0}^{r} d_i \alpha_i$$

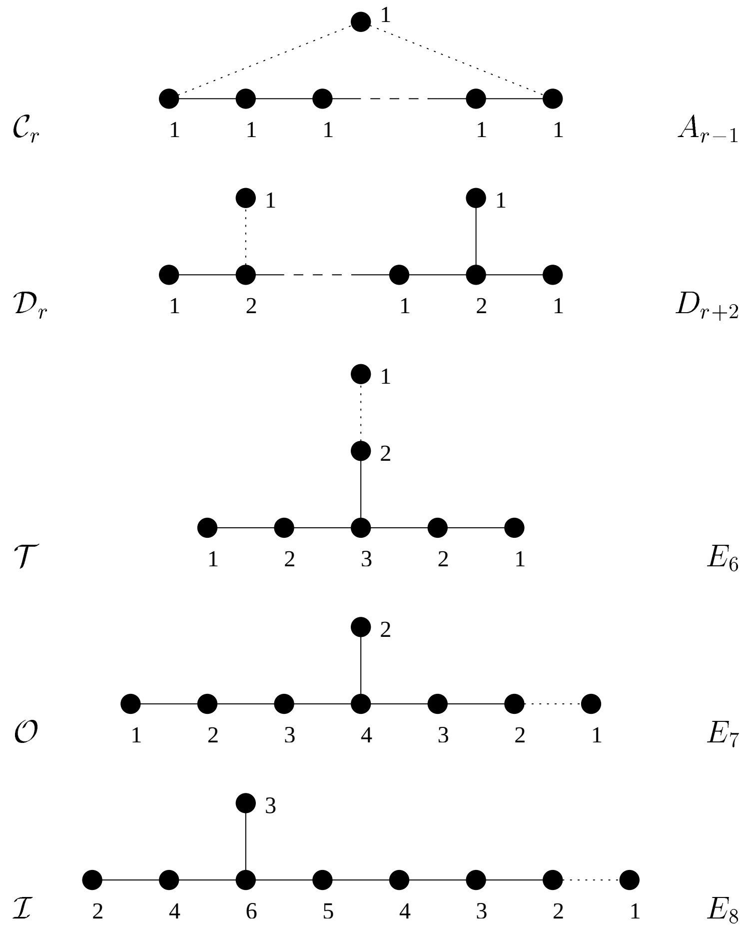



Table 6: The McKay correspondence

In the diagrams we have added to each node the corresponding coefficient $d_i$. There is the following result relating these coefficients to the dimensions of the irreducible representations:

**Proposition 6.4.** *Let the $d_i$ and $\rho_i$ be as above and let $h$ be the Coxeter number of the Lie algebra* $\mathbf{g}$. *Then the following holds for all $j = 0, ..., r$:*

$$dim(\gamma_i) = d_i \ . \qquad and \quad h = \sum_{i=0}^{r} dim(\rho_i)$$

*Proof.* Let $\mathbf{d} = \{d_i\}$ and $\mathbf{e} = \{dim(\rho_i)\}$ be vectors in $\mathbb{R}^{r+1}$. We want to show that they are equal. We will first show they are both eigenvectors of eigenvalue 2 for the matrix $A(\Gamma)$. This is the case for $\mathbf{e}$, since $\mathbf{e} = \{\chi_j(id)\}$, which has eigenvalue 0 for $C = 2I - A(\Gamma)$, by proposition 6.2. The vector $\mathbf{d}$ is orthogonal to every vector in $\widetilde{\mathbf{h}}'$, hence $\widetilde{\mathcal{W}}$-invariant. This gives us

$$2d_i = \sum_{j \in N(i)} d_j \tag{6}$$

so also $\mathbf{d}$ is eigenvector of eigenvalue 2. The space spanned by eigenvectors of $A(\Gamma)$ of eigenvalue 2, is one dimensional since they are eigenvectors of the extended Cartan matrix, of eigenvalue 0; this $r+1 \times r+1$-matrix contains the non singular $r \times r$-matrix $C(\mathbf{g})$ as a principal minor. The fact that $d_0 = 1 = dim(\rho_0) = e_0$ concludes the proof of the first statement.

Now the second follows from the well-known fact that $h$ is the sum of the coefficients of the highest roots plus one. $\square$

If we consider the numbers $d_i = dim(\rho_i)$ to be the image of a function $f$ of the nodes of the graph to the strictly positive integers $\mathbb{Z}_{>0}$, the equality in (6) is the condition of $f$ to be an additive function of the graph. It can be shown that the existence of such an additive function on a connected graph with no loops is actually defining for it to be an extended Dynkin diagram of type $A$, $D$, or $E$. See [30].

We would like to mention some results obtained by using the McKay correspondence. A good example of the use of it is illustrated in the article of Kostant, see [22]. In this article he attacks the following question: Let $S = \bigoplus_{n=0}^{\infty} S^n(\mathbb{C}^2)$ be the symmetric algebra over $\mathbb{C}^2$ and let $\pi_n$ be the representation of $SU_2(\mathbb{C})$ on the $n$-th component. These representations are irreducible for all $n$ and set up a basis for the unitary dual of $SU_2(\mathbb{C})$. Given a finite subgroup $\Gamma \subset SU_2(\mathbb{C})$, we want to see how the restriction of $\pi_n$ to $\Gamma$ decomposes.

Now Kostant shows that the root-structure of the Lie algebra that is associated to $\Gamma$ contains all the information about the multiplicities of the decomposition of the representations mentioned above. These multiplicities come in a beautiful way from the orbit structure of the Coxeter element on the roots of $\mathbf{g}$.

The results obtained by Kostant in this paper were proven later in a slightly different way, however with the use of the McKay correspondence, by Springer, see [39].

The observation made by McKay and the ideas discussed above have been a starting point for many others, either using the McKay correspondence to get new results, as for instance Kostant in the article mentioned above, or trying to give an explanation or generalization of it. One of these directions is discussed in the next section.

# 7 Geometric McKay correspondence

We have seen that to each finite $\Gamma$ in $SU_2(\mathbb{C})$ we can associate a Dynkin graph, the resolution graph of the quotient singularity. On the other hand the McKay correspondence gives a correspondence between the affine version of these graphs and the irreducible representations of $\Gamma$. The question now rises whether these representations and the resolution of the singularity are more directly related.

This is indeed the case. Gonzalez-Springberg and Verdier have proven a theorem which states such a connection, see [10]. There is another proof of this by H. Knörrer [19], which is more accessible. This connection is sometimes referred to as the 'geometric' McKay correspondence, where the McKay correspondence discussed in the last section is called 'algebraic'.

In short the theorem states there is a bijection between the irreducible representations of a binary polyhedral group and irreducible divisors of the minimal resolution of the corresponding quotient singularity.

Let $\Gamma$ be binary polyhedral, $V := \mathbb{C}^2$, $S := V/\Gamma$ and $\phi : \tilde{S} \to S$ the minimal resolution with exceptional set $D$. Let $\rho : \Gamma \to GL(E)$ be an irreducible representation of $\Gamma$. There is a natural action of $\Gamma$ on $V$, that makes it possible to define the $\Gamma$-equivariant vector bundle $\mathcal{E} := V \times E \to V$, on which $\Gamma$ acts by $\gamma.(\mathbf{z}, \mathbf{v}) = (\gamma(\mathbf{z}), \gamma(\mathbf{v}))$.

Now the associated locally free sheaf on $V$ is equal to $\mathcal{O}_V \otimes_{\mathbb{C}} E$. It has a canonical $\Gamma$-action. We denote it, by abuse of notation, with $\mathcal{E}$.

The group $\Gamma$ acts freely on $V - \{0\}$ and $\mathcal{E}$ is a $\Gamma$-vector bundle, hence $\mathcal{E}$ defines a vector bundle $\mathcal{E}'$ on the quotient $S - \{0\} = (V - \{0\})/\Gamma$. Let $\tilde{\mathcal{E}} := \phi^*(\mathcal{E}')$ be the pull-back of this on $\tilde{S} - D \cong S - \{0\}$. Let $i : \tilde{S} - D \to \tilde{S}$ be the inclusion map. If $s$ is an invariant global section of $\mathcal{E}$, $s \in \Gamma(V, \mathcal{E})^\Gamma$, then $s$ induces a global section on $\mathcal{E}'$ and $\tilde{\mathcal{E}}$. So $s$ defines a section $\pi(s)$ of the sheaf $i_*(\tilde{\mathcal{E}})$ on $\tilde{S}$. Denote the subsheaf of $i_*(\tilde{\mathcal{E}})$ generated by the sections $\pi(s)$, $s \in \Gamma(V, \mathcal{E})^\Gamma$, by $\pi(\rho)$. By $c_1(\pi(\rho))$ is meant the first Chern class of the sheaf $\pi(\rho)$. Let $\mathrm{Irr}(D)$ be the set of irreducible components of $D$ and let $\widehat{\Gamma}$ be the set of non-trivial irreducible representations. By $a_{ij}$ we denote the coefficient of $\rho_j$ in the decomposition of $\rho \otimes \rho_i$, this $\rho$ being the canonical two-dimensional representation of $\Gamma$.

**Theorem 7.1.** *For each $\rho \in \widehat{\Gamma}$ the sheaf $\pi(\rho)$ on $\tilde{S}$ is locally free of deg($\rho$). There is a bijection $\psi : \widehat{\Gamma} \to Irr(D)$ such that*

$$c_1(\pi(\rho)) \cdot d = \begin{cases} 0 & \textit{if } d \neq \psi(\rho) \\ 1 & \textit{if } d = \psi(\rho) \end{cases}$$

*and*

$$\psi(\rho_i) \cdot \psi(\rho_j) = a_{ij} \textit{ for all } i \neq j.$$

So for each non trivial irreducible representation $\rho_i$, a locally free sheaf $\pi(\rho_i)$ on $\tilde{S}$ is constructed, which by its first Chern class is represented by a divisor. This divisor meets the exceptional set $D$ transversally in one point, hence in a unique component $D_i \in D$, which is taken to be the image of $\rho_i$ under $\psi$.

# 8 Brieskorn's Theorem

## 8.1 Introduction

In this section we discuss a theorem by Brieskorn, which establishes a direct connection between the binary polyhedral groups with their corresponding singularities and the Lie algebras of the type of the Dynkin diagrams obtained in the resolution of the singularities. We will see that the nilpotent variety of such a Lie algebra intersected with a slice transversal to the subregular nilpotent orbit is a singularity with the Dynkin diagram of this Lie algebra as its resolution graph.

At many points there are generalizations possible, but we do not want to explore in this direction. The goal here is to demonstrate the statement above in the simplest case.

The results presented in this section have a history in the studies of the connection between quotient singularities and Dynkin diagrams, presented in section 5, and the work by Steinberg, Kostant and Springer on conjugacy classes and the unipotent variety of an algebraic group. Brieskorn studied further connections between the resolutions of singularities and structures related to their Dynkin diagrams, such as the Weyl group and Coxeter transformations. That led Grothendieck to conjecture the above mentioned connection between the simple singularities and Lie algebras or groups of type $A$, $D$, $E$.

We want to mention here the fundamental papers by Kostant [20], [21], Steinberg [41], and Springer [37].

To start with we take a complex simple algebraic group $G$. We let $G$ act on itself by the conjugation action:

$$\begin{array}{rcl} G \times G & \longrightarrow & G \\ (g,h) & \mapsto & ghg^{-1} \ . \end{array}$$

This action is called the adjoint action. We have the linearization of this action, the adjoint representation of $G$ on the Lie algebra of $G$, $\mathbf{g}$:

$$\begin{array}{rcl} G \times \mathbf{g} & \longrightarrow & \mathbf{g} \\ (g,h) & \mapsto & ad(g)(h) \ . \end{array}$$

Let $T$ be a maximal torus of $G$ with character group $X^*(T) \cong \mathbb{Z}^r$, so $\mathrm{rank}(G) = \mathrm{rank}(\mathbf{g}) = r$. Let $N = N_G(T)$ be the normalizer of $T$ in $G$, leading us to the Weyl group of $G$, $\mathcal{W} = N/T$. If we restrict the adjoint representation to $T$ we have the decomposition of $\mathbf{g}$ into a direct sum of eigenspaces:

$$\mathbf{g} = \bigoplus_{\alpha \in X^*(T)} \mathbf{g}_\alpha \ .$$

This defines the set of roots of $T$ in $\mathbf{g}$. By choosing a Borel subgroup of $G$, containing $T$, we make a choice for a set of positive roots and a corresponding basis of simple roots, $\Delta := \{\alpha_1, ..., \alpha_r\}$, and we have the fundamental dominant weights $\lambda_i$, $i = 1, ..., r$, defined by

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \ ,$$

$(\ ,\ )$ being the $\mathcal{W}$-invariant scalar product on the $r$-dimensional Euclidean space, spanned by the roots.

We recall that for each of these fundamental dominant weights we have an irreducible highest weight representation of $G$, $\rho_\lambda$, with character $\chi_\lambda \in \mathbb{C}[G]$, which is constant on conjugacy classes of $G$.

We have a Jordan decompositions in $G$ and $\mathbf{g}$. In $G$ it realizes a decomposition of each $x \in G$ in its semisimple and unipotent part:

$$x = x_s \cdot x_u,\ x_s \text{ semisimple, } x_u \text{ unipotent , such that } x_s \text{ and } x_u \text{ commute.}$$

In the Lie algebra it decomposes each $x \in \mathbf{g}$ in its semisimple and nilpotent part:

$$x = x_s + x_n,\ x_s \text{ semisimple, } x_n \text{ nilpotent, with } [x_s, x_n] = 0.$$

These decompositions are unique and representations respect them.

## 8.2 The adjoint quotient

Let us consider the adjoint representation of $G$ on the Lie algebra of $G$, $\mathbf{g}$. We want to study the orbit structure, i.e. look for a quotient map. This was done extensively by Kostant in his papers mentioned above.

Let $\mathbf{h}$ be the Lie algebra of the maximal torus $T$. On it we have a natural action of $\mathcal{W}$. We want to study the ring of $G$-invariant polynomials in $\mathbf{g}$, $\mathbb{C}[\mathbf{g}]^G$. We have the following result.

**Theorem 8.1.** *The algebras $\mathbb{C}[\mathbf{g}]^G$ and $\mathbb{C}[\mathbf{h}]^{\mathcal{W}}$ are isomorphic. There exist $r$ homogeneous $G$-invariant polynomials $\gamma_i$ of degrees $d_i = m_i+1$, that generate $\mathbb{C}[\mathbf{g}]^G$ freely as a $\mathbb{C}$-algebra. The $m_i$ are the exponents of the Weyl group of* $\mathbf{g}$.

*Proof.* Consider the mapping

$$\mathbb{C}[\mathbf{g}] \to \mathbb{C}[\mathbf{h}],$$

realized by restricting polynomials to the subalgebra $\mathbf{h}$. Its restriction to the algebra of invariants $\mathbb{C}[\mathbf{g}]^G$ gives an isomorphism with $\mathbb{C}[\mathbf{h}]^{\mathcal{W}}$. It is a result due to Chevalley that the last is freely generated as a $\mathbb{C}$-algebra by $r$ homogeneous polynomials $p_i$ of degrees $d_i = m_i + 1$, the $m_i$ being the exponents of the Weyl group of $\mathbf{g}$. For a complete proof see [4], paragraph 8, no. 3, Th. 1 and Cor. 1. ☐

The semisimple conjugacy classes in $\mathbf{g}$, under the adjoint action of G, are in natural correspondence with elements of $\mathbf{h}_{/\mathcal{W}}$; $s, t \in \mathbf{h}$ are conjugated under $G$ if and only if they are in the same $\mathcal{W}$-orbit. Furthermore the $\gamma_i$'s separate semisimple orbits: Let $s, t \in \mathbf{g}$ be semisimple. Then

$$\begin{aligned}
&s \text{ and } t \text{ conjugate under } G \iff \\
&s \text{ and } t \text{ conjugate under } \mathcal{W} \iff \\
&\gamma_i(s) = \gamma_i(t) \quad \forall i \in \{1, \ldots, r\}
\end{aligned}$$

The embedding of $\mathbb{C}[\mathbf{h}]^{\mathcal{W}}$ into $\mathbb{C}[\mathbf{g}]$, induces a morphism $\mathbf{g} \to \mathbf{h}/\mathcal{W}$, called the adjoint quotient of $\mathbf{g}$;

$$\gamma : \mathbf{g} \to \mathbf{h}/\mathcal{W} \ .$$

It can be realized by generators $\gamma_i$ of the algebra of invariants, thus taking a concrete form and making $\mathbf{h}/\mathcal{W}$ isomorphic to affine $r$-dimensional complex space.

$$\begin{aligned}
\gamma : \mathbf{g} &\to \mathbb{C}^r \\
x &\mapsto (\gamma_1(x), ..., \gamma_r(x))
\end{aligned}$$

We now want to study the fibers of the adjoint quotient. These fibers must be unions of $G$-orbits in $\mathbf{g}$. Let us collect some results on the adjoint quotient and its fibers.

**Proposition 8.2.** *Let $\gamma : \mathbf{g} \to \mathbf{h}/\mathcal{W}$ be the adjoint quotient defined above. Then we have the following:*

1. *$\gamma$ is a flat morphism; all fibers of $\gamma$ have dimension dim($\mathbf{g}$)$-r$.*
2. *Each fiber is a union of finitely many $G$-orbits in* $\mathbf{g}$.
3. *The orbit dimensions have to be even.*

*Proof.* For a proof see [21]. ☐

## 8.3 Regular and subregular elements, and the nilpotent variety

The elements for which the orbit dimension is maximal are called regular. The orbit dimension is maximal if the dimension of the stabilizer subgroup, $C_G(x)$, is minimal, $C_G(x) := \{g \in G | g.x = x\}$ for $x \in \mathbf{g}$. The minimal dimension of $C_G(x)$ is equal to the rank of $\mathbf{g}$, $r$. It can be shown that the Lie algebra of $C_G(x)$, $\mathbf{c}_G(x)$, contains an $r$-dimensional commutative subalgebra.

This gives us that if $O$ is an orbit that

$$dim(O) \leq dim(G) - r.$$

So if an orbit is not regular, it has to have dimension at least 2 less then $\dim(G) - r$, the dimension of the centralizer $C_G(x)$ is $r + 2i$, $i = 1, 2, ...$ Irregular elements for which the centralizer has maximal dimension, that is dimension $r + 2$, are called subregular.

**Proposition 8.3.** *Each fiber contains a unique regular orbit, which is dense in it.*

*Proof.* See [21]. □

**Proposition 8.4.** *An element $x \in \mathbf{g}$ is nilpotent $\iff$ $\gamma(x) = \gamma(0)$.*

*Proof.* This is a special case of the theorem that for $x \in \mathbf{g}$, and $f \in \mathbb{C}[\mathbf{g}]^G$, $f(x) = f(x_s)$. For a proof of this more general statement see [42]. □

This allows us to define the nilpotent variety,

$$Nil(\mathbf{g}) := \{x \in \mathbf{g} | \gamma(x) = \gamma(0)\} = \gamma^{-1}(\gamma(0)).$$

**Proposition 8.5.** *There is a unique subregular nilpotent $G$-orbit in* $\mathbf{g}$.

*Proof.* See [42], 3.10 Th. 1. □

## 8.4 Example: $\mathbf{sl}_n$

To illustrate the theory of the last sections we look at the case $G = SL_{r+1}$. Then the Lie algebra $\mathbf{sl}_{r+1}$ is given by the $r+1 \times r+1$-matrices of trace zero. The adjoint representation of $G$ is given by conjugation:

$$\begin{array}{rcl} G \times \mathbf{g} & \to & \mathbf{g} \\ (g, x) & \mapsto & gxg^{-1} \end{array}$$

If we choose $T$ the set of diagonal matrices in $G$, we get the corresponding Cartan subalgebra $\mathbf{h}$ the diagonal matrices in $\mathbf{g}$. It has dimension $r$. The

Weyl group $\mathcal{W}$ is isomorphic to $S_{r+1}$, the symmetric group on $r+1$ elements. It acts on $T$ and $\mathbf{h}$ by permuting the entries.

The adjoint quotient $\gamma$ can be realized by the map

$$\begin{array}{rcl} \gamma : \mathbf{sl}_{r+1} & \to & \mathbb{C}^r \\ x & \mapsto & (\gamma_1(x), ..., \gamma_r(x)), \end{array}$$

such that

$$det(\lambda \cdot Id - x) = \lambda^{r+1} + \gamma_1(x)\lambda^{r-1} + ... + \gamma_r(x) \ ,$$

is the characteristic polynomial of $x$.

Let us take a look at the fibers of $\gamma$. Take $c = (c_1, ..., c_r) \in \mathbb{C}^r$. It defines a polynomial $p(c) = \lambda^r + c_1\lambda^{r-1} + ... + c_r$, which in turn defines a partition $m(c) = (m_1, .., m_k)$ of $r+1$ by taking the multiplicities of the roots, i.e. the eigenvalues, of $p$, say $t_1, ..., t_k$. We take all partitions decreasing. Each $m_i$ is the total size of the blocks with one eigenvalue, $t_i$; there are partitions $s_i$ of the $m_i$, $i = 1, ..., k$, $s_i = (s_{i1}, ..., s_{il_i})$. In the Jordan normal form of $x$, $s_{ij}$ is the size of the $j$-th block for the $i$-th eigenvalue $t_i$. Each $x \in \chi^{-1}(c)$ is up to conjugacy determined by such a sequence $s_1, ..., s_k$ of partitions of the $m_i$. Let us illustrate this by putting $x$ into Jordan normal form:

$$\begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_k \end{pmatrix}$$

with each $M_i$ a $m_i \times m_i$-matrix with all eigenvalues $t_i$ and Jordan form

$$M_i = \begin{pmatrix} S_{i1} & & & \\ & S_{i2} & & \\ & & \ddots & \\ & & & S_{il_i} \end{pmatrix}, \quad S_{ij} = \begin{pmatrix} t_i & 1 & & \\ & t_i & \ddots & \\ & & \ddots & 1 \\ & & & t_i \end{pmatrix},$$

an $s_{ij} \times s_{ij}$-matrix.

So clearly there are finitely many orbits in each fiber of $\gamma$. We see that

- $x$ is semisimple exactly when all the $S_{ij}$ are one-dimensional, the partitions $s_i$ are $(1, 1, ..., 1)$.
- $x$ is regular exactly when $s_i = (m_i) \forall i = 1, ..., k$
- $x$ is subregular exactly when $s_i = (m_i) \forall i = 1, ..., k$ but one, say $j$, for which $s_j = (m_j - 1, 1)$.

If we restrict to $x$ nilpotent, so $k = 1$, we see that for $x$ to be semisimple, regular, subregular, it means that $x$ is conjugate to

$$\begin{pmatrix} & & \\ & 0 & \\ & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

respectively. It is clear that the regular and the subregular nilpotent orbits are unique.

$$\begin{array}{cc} \text{partition} & \text{dim} \\ \hline (5) & 20 \\ | & | \\ (4,1) & 18 \\ | & | \\ (3,2) & 16 \\ | & | \\ (3,1,1) & 14 \\ | & | \\ (2,2,1) & 12 \\ | & | \\ (2,1,1,1) & 8 \\ | & | \\ (1,1,1,1,1) & 0 \end{array} \qquad \begin{array}{cc} \text{partition} & \text{dim} \\ \hline (6) & 30 \\ | & | \\ (5,1) & 28 \\ | & | \\ (4,2) & 26 \\ / \quad \backslash & / \ \backslash \\ (4,1,1) \quad (3,3) & 24 \quad 24 \\ \backslash \quad / & \backslash \ / \\ (3,2,1) & 22 \\ / \quad \backslash & / \ \backslash \\ (3,1,1,1) \quad (2,2,2) & 18 \quad 18 \\ \backslash \quad / & \backslash \ / \\ (2,2,1,1) & 16 \\ | & | \\ (2,1,1,1,1) & 10 \\ | & | \\ (1,1,1,1,1,1) & 0 \end{array}$$

An interesting question is how the nilpotent variety decomposes into $G$-orbits. We know there is one unique dense orbit, the regular orbit, and that every other orbit has smaller dimension. The semisimple orbit, which is a single point lies in the closure of every other orbit in $Nil(\mathbf{g})$. We can define the closure relation between arbitrary orbits in $\mathbf{g}$. Let $O$ and $O'$ be two nilpotent orbits, defined by partitions $\pi = (n_1, ..., n_{l'})$ and $\pi' = (n'_1, ..., n'_{l'})$. We define the partial ordering on the set of orbits by

$$O \leq O' \iff O \text{ is contained in } \overline{O'}.$$

This is the partial order on partitions

$$O \leq O' \iff \pi \leq \pi' \iff n_1 + n_2 + \cdots + n_i \leq n'_1 + n'_2 + \cdots + n'_i \quad \forall i.$$

We have illustrated this, for r=5 and r=6.

## 8.5 Brieskorn's theorem

The subregular nilpotent orbit has a particular property. Let us first make the following definition:

**Definition 8.6.** *A transversal slice to the $G$-orbit of an element $x \in \mathbf{g}$ in the point $x$ is a smooth locally closed subvariety $S \subset \mathbf{g}$, such that $x \in S$, dim(S) = codim(Gx), and the map $G \times S \to \mathbf{g}$ is a submersion.*

In our case we are interested in a slice transversal to the orbit of a subregular nilpotent element $x \in \mathbf{g}$. This is a subspace $S \subset \mathbf{g}$, such that $x \in S$ and $S$ is complementary to the affine tangent space at $x$ of the orbit $Gx$.

Now we can state the main theorem we want to illustrate in this section, conjectured by Grothendieck, and later on proved by Brieskorn:

**Theorem 8.7.** *Let $x$ be a subregular nilpotent element of a simple complex Lie algebra $\mathbf{g}$ of type $\Delta = A_r$, $D_r$, or $E_r$, and let $S$ be a transversal slice to the $G$-orbit of $x$ in the point $x$. Then*

1. *$S \cap Nil(\mathbf{g})$ is a surface with a Kleinian singularity of type $\Delta$.*
2. *The restriction $\gamma_{|S} : S \to \mathbf{h}/\mathcal{W}$ is a semi-universal deformation of the singularity $(S \cap Nil(\mathbf{g}), x)$.*

*Proof.* The possible outlines of a proof, for the first part, were given by Brieskorn, see [6]. For a complete proof of this theorem see Slodowy [32]. It gives a thorough account of this result, making generalizations where possible. For a proof of the second part of the proof the reader is referred to this book. In what follows we will discuss the first part. □

## 8.6 Identification of the subregular singularity

There are two different strategies to get the identification of the singularities. The first is by giving a resolution of the singularity. The second is by giving a $\mathbb{C}^*$-action on $S$ and on $\mathbf{h}/\mathcal{W}$, such that the restriction of $\gamma$ to $S$ becomes $\mathbb{C}^*$-equivariant, that is weighted homogeneous of the same type as the polynomial $R$ in table 4. Then the identification follows from proposition 4.2.

We will sketch the outlines of the second method, and conclude with some remarks on the first. It is of great interest and historically the first viewpoint.

Let us recall the following result:

**Lemma 8.8.** *(Jacobson-Morozov) Let* $\mathbf{g}$ *be a reductive Lie algebra and let* $x \in \mathbf{g}$ *be nilpotent. Then there exist* $y, h \in \mathbf{g}$ *and an injective homomorphism of Lie algebras,*

$$\rho : \mathbf{sl}_2 \longrightarrow \mathbf{g} \ ,$$

*with* $\rho(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = x$*,* $\rho(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = h$*, and* $\rho(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = y$*. This gives us the relations:*

$$[h, x] = 2x \ , \ [h, y] = -2y \ , \ and \ [x, y] = h.$$

Let us take $x \in \mathbf{g}$ nilpotent. The tangent space to the orbit of $x$ is given by

$$T_x(Gx) = x + [\mathbf{g}, x].$$

We want to take a transversal slice $S$ to the $G$-orbit of $x$ in $x$. This has to be of the form

$$S := \{s \in \mathbf{g} | s = x + e, \ e \text{ in the linear complement of } [\mathbf{g}, x]\}$$

To determine $[\mathbf{g}, x]$ and a linear complement we now use the Jacobson-Morozov lemma to get $y, h \in \mathbf{g}$, such that $A :=< x, h, y >$ is a $\mathbf{sl}_2$-subalgebra of $\mathbf{g}$. $\mathbf{g}$ becomes an $A$-module by the adjoint representation of $\mathbf{g}$. We have a complete description for such $\mathbf{sl}_2$-modules. They are the direct sum of irreducible $n+1$-dimensional $\mathbf{sl}_2$-modules $V_{n+1}$, which with respect to $h \in A$, are the direct sum of one-dimensional subspaces $V_{n+1}(\lambda)$, $\lambda = -n, -n+2, ..., n-2, n$. Here $\lambda$ is the eigenvalue of $h$ acting on $V_{n+1}$.

So we get a string $\{n_i\} \in \mathbb{Z}^k_{\geq 0}$, such that $\mathbf{g}$ splits

$$\mathbf{g} = \bigoplus_{i=1}^{k} V_{n_i+1}$$

into irreducible submodules, as an $A$-module. And we see that

$$[\mathbf{g}, x] = ad(x)(\mathbf{g}) = \bigoplus_{i=1}^{k} ad(x)(V_{n_i+1}).$$

Since in an $A$-module $V_{n_i+1}$, $ad(x)$ maps $V_{n_i+1}(k)$ onto $V_{n_i+1}(k+2)$, for $k \geq -n$, we see that in $V_{n_i+1}$ a complement to $ad(x)(V_{n_i+1})$ is spanned by the vector of weight $-n_i$, i.e. the space $S_i := V_{n_i+1}(-n_i)$. We see that a linear complement to $[\mathbf{g}, x]$ is given by $\bigoplus_{i=1}^{k} S_i$. This is exactly the centralizer of the element $y \in \mathbf{g}$. It has dimension equal to the dimension of the centralizer of $x$; we see that for $x$ regular nilpotent, $k = r$, and for $x$ subregular nilpotent

$k = r + 2$. We have $S = x + \bigoplus_{i=1}^{k} S_i$ is a transversal slice to the $G$-orbit of $x$ in the point $x$.

Now that we have a transversal slice to the orbit of nilpotent $x \in \mathbf{g}$ we introduce an action of $\mathbb{C}^*$ on it. This action will have to stabilize $S$, so that the restriction of $\gamma$ to $S$ can be made equivariant with respect to this action.

Firstly we have the action of scalar multiplication on $\mathbf{g}$, which we denote by $\sigma_1$:

$$\begin{array}{rcl} \mathbb{C}^* \times \mathbf{g} & \to & \mathbf{g} \\ (t, x) & \mapsto & \sigma_1(t)x = tx. \end{array}$$

We have another action of $\mathbb{C}^*$, with the use of our Jacobson-Morozov homomorphism constructed above. The subalgebra $< h > \subset \mathbf{sl}_2$ corresponds to a multiplicative group $\mathbb{C}^*$ in $T \subset G$, so to action of $h$ we can associate an action of $\mathbb{C}^*$. Let us take vectors $s_i \in S_i$. Now an element of $s \in S$ can be written as $s = x + \sum_{i=1}^{k} c_i s_i$. The action mentioned above on such a vector is as follows:

$$(t, s) \mapsto \sigma_2(t)(x + \sum_{i=1}^{k} c_i s_i) = t^2 x + \sum_{i=1}^{k} t^{-n_i} c_i s_i.$$

Neither of the actions $\sigma_1$ and $\sigma_2$ stabilizes the subspace $S$. An easy computation shows that these actions on $S$ commute. This allows us to consider a certain product of these actions, namely $\sigma(t) = \sigma_1(t^2)\sigma_2(t^{-1})$, acting on a vector $s \in S$ by

$$\sigma(t)(x + \sum_{i=1}^{k} c_i s_i) = x + \sum_{i=1}^{k} t^{n_i+2} c_i s_i.$$

The action $\sigma$ does stabilize the subspace $S$, and with respect to the coordinate system $\{s_i\}$ it is linear with weights $n_i + 2$.

As we have stated above the adjoint quotient $\mathbf{g} \to \mathbf{h}/\mathcal{W} \cong \mathbb{C}^r$ can be realized by the $\gamma_i$, these being $G$-invariant homogeneous of degree $m_i + 1$. So we have

$$\gamma_i(\sigma_1(t)v) = t^{m_i+1}\gamma_i(v) \quad \forall i \in \{1, ..., r\},$$

by homogeneity and

$$\gamma_i(\sigma_2(t)v) = \gamma_i(v) \quad \forall i \in \{1, ..., r\},$$

by $G$-invariance.

We want to specify a $\mathbb{C}^*$-action on $\mathbb{C}^r$ that makes the restriction of $\gamma = (\gamma_1, ..., \gamma_r)$ to $S$ $\mathbb{C}^*$-equivariant, having the $\sigma$-action on $S$. On a vector $s = x + \sum_{i=1}^{k} c_i s_i$ we have

$$\gamma_j(\sigma(t)s) = \gamma_j(\sigma_1(t^2)\sigma_2(t^{-1})s) = \gamma_j(\sigma_1(t^2)s) = t^{m_j+2}\gamma_j(x).$$

| | $d_1$ | $d_2$ | $d_3$ | $\cdots$ | $\cdots$ | $d_{r-2}$ | $d_{r-1}$ | $d_r$ | $w_r$ | $w_{r+1}$ | $w_{r+2}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_r$ | 4 | 6 | 8 | $\ldots$ | $\ldots$ | $2r-2$ | $2r$ | $2r+2$ | 2 | $r+1$ | $r+1$ |
| $D_r$ | 4 | 8 | 12 | $\ldots$ | $\ldots$ | $4r-8$ | $2r$ | $4r-4$ | 4 | $2r-4$ | $2r-2$ |
| $E_6$ | 4 | 10 | 12 | | | 16 | 18 | 24 | 6 | 8 | 12 |
| $E_7$ | 4 | 12 | 16 | | 20 | 24 | 28 | 36 | 8 | 12 | 18 |
| $E_8$ | 4 | 16 | 24 | 28 | 36 | 40 | 48 | 60 | 12 | 20 | 30 |

Table 7: Weights and degrees for the adjoint quotient.

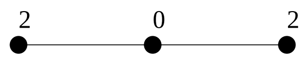
So we get the desired result if we let $\mathbb{C}^*$ act on $\mathbf{h}/\mathcal{W} \cong \mathbb{C}^r$ by the weights $d_j := 2m_j+2$. Then $\gamma$ becomes weighted homogeneous of type $(d_1, ..., d_r; n_1+2, ..., n_k+2)$.

As stated before in the case that $x$ is a regular element, the number of irreducible $A$-modules must be equal to the rank of $\mathbf{g}$, $k = r$. If we consider the restriction of $\gamma$ to $S$ it becomes weighted homogeneous of type $(2m_1+2, .., 2m_r+2; n_1+2, ..., n_r+2)$. It is a well-known fact that in this case the degrees and the weights are equal, hence $2m_i = n_i$, for all $i = 1, ..., r$. In fact this equality of weights and degrees can be used to compute the exponents $m_i$, from the weights $n_i$.

Varadarajan [45] showed that this result together with the linear independence of the differentials $d\gamma_i$ in all regular elements of $\mathbf{g}$, follows from a more general formula by Harish-Chandra [11] on differential operators on $\mathbf{g}$.

For subregular nilpotent $x$ we have listed the weights $w_1, ..., w_{r+2}$ and the degrees $d_1, ..., d_r$ in table 7. The computation of the values $d_i$ and $w_i$ shows that the first $r-1$ values of the weights and the degrees are equal; $d_i = w_i$ for $i = 1, ..., r-1$, so we have not listed the first $r-1$ values of $w_i$. We get the $d_i$ from the exponents of $\mathbf{g}$. They can be found in the tables in Bourbaki, see [3], Planches. The $n_i$ are the highest weights of $\mathbf{g}$ as the particular $\mathbf{sl}_2$-module constructed above. The values $\alpha(h)$, for the simple roots $\alpha$ can be taken from the valuated Dynkin diagram of the subregular nilpotent $x$. Then all the eigenvalues of $ad(h)$ on $\mathbf{g}$ can be calculated. This gives the values $n_i$. See [9] for tables.

As an example we consider the subregular class in the simple Lie algebra of type $A_3$. The valuated Dynkin diagram then has the form




where the nodes correspond from left to right to the simple roots $\alpha_1$, $\alpha_2$ and $\alpha_3$. The valuation is realized by the eigenvalue of $ad(h)$ on the root space

$\mathbf{g}_{\alpha_i}$. The set of positive roots, $\Pi$, is given by:

$$\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

On $\mathbf{n}^+ := \bigoplus_{\alpha \in \Pi} \mathbf{g}_\alpha$ $ad(h)$ acts with eigenvalues 2, 0, 2, 2, 2, 4 respectively. On $h$ it has three times eigenvalue 0, and on $\mathbf{n}^-$ it acts with eigenvalues $-2$, 0, $-2$, $-2$, $-2$, $-4$. So $\mathbf{g}$ splits into five irreducible $\mathbf{sl}_2$-modules, one of highest weight 0, three of highest weight 2 and one of highest weight 4. This gives the values 2, 4, 4, 4, 6. for the weights of the restriction $\gamma_{|S}$.

We are almost ready to identify the surface singularity $Nil(\mathbf{g}) \cap S$ as a Kleinian singularity. There is one more lemma we need.

**Lemma 8.9.** *The restriction of the adjoint quotient to the transversal slice $S$, $\chi_{|S} \to \mathbf{h}/\mathcal{W}$, has rank $r-1$ in $x$.*

*Proof.* This is Lemma 8.3.1 in [32]. ☐

Using the lemma and the fact that $w_i = d_i$ for $i = 1, ..., r-1$, we may suppose that the adjoint quotient $\chi_{|S} : S \to \mathbf{h}/\mathcal{W}$ is of the form

$$\begin{array}{rl} S = M \oplus B & \longrightarrow E \oplus B \\ (a, b) & \mapsto (\delta(a, b), p(b)) \end{array}$$

where dim$(B)$= $r-1$, dim$(M)$= 3, dim$(E)$= 1, $p$ is a polynomial isomorphism and $\delta$ the unique singular component weighted homogeneous of type $(d_r; w_r, w_{r+1}, w_{r+2})$. Hence the singularity $S \cap Nil(\mathbf{g})$ can be defined as an hypersurface in $M$ by a polynomial $f$, weighted homogeneous of type $(d_r; w_r, w_{r+1}, w_{r+2})$. Comparing these values to the values in table 4 and using proposition 4.2 this $f$ can be taken to have the following normal form:

$$\begin{array}{ll} A_r & X_{r+1} + YZ \\ D_r & X_{r-1} + XY^2 + Z^2 \\ E_6 & X^4 + Y^3 + Z^2 \\ E_7 & X^3Y + Y^3 + Z^2 \\ E_8 & X^5 + Y^3 + Z^2 \end{array}$$

As we have seen these polynomials are defining for the Kleinian singularities. This concludes the proof of the first part of the theorem.

Let us now illustrate Brieskorn's theorem with an example. We take $\mathbf{g}$ of type $A_3$, that is $\mathbf{g} = \mathbf{sl}_4$. For the subregular nilpotent element $x$ we take

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This element lies in the $\mathbf{sl}_2$-triple $\{x, h, y\} \subset \mathbf{g}$, with

$$h = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix} y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

An element in a transversal slice $S$ to the orbit of $x$ is of the form $s = x+c$, with $c$ in the centralizer of $y$. This leads to the following:

$$S = \{ \begin{pmatrix} x & 1 & 0 & 0 \\ v & x & 1 & 0 \\ w & 3v & x & y \\ z & 0 & 0 & -3x \end{pmatrix} |(v, w, x, y, z) \in \mathbb{C}^5 \}$$

The characteristic polynomial of an element $s \in S$ is

$$\begin{aligned} p(v, w, x, y, z, \lambda) = \lambda^4 \quad & +(-6x^2 - 4v)\lambda^2 + (-8vx - 8x^3 - w)\lambda \\ & +(-3x^4 - zy + 12vx^2 - 3wx) \end{aligned}$$

so the restriction of the adjoint quotient to $S$ is the map

$$\begin{aligned} \chi : S \quad & \to \mathbb{C}^3 \\ (v, w, x, y, z) \quad & \mapsto (-6x^2 - 4v, -8vx - 8x^3 - w, -3x^4 - zy + 12vx^2 - 3wx) \end{aligned}$$

after substituting $v = (-u_1 + 6x^2)/4$ and $w = 20x^3 + 2u_1x - u_2$ we get

$$\chi(u_1, u_2, x, y, z) = (u_1, u_2, -81x^4 - yz - 9u_1x^2 + 3u_2x)$$

which is in fact the semiuniversal deformation of a singularity of type $A_3$. That is the second statement of Brieskorn's theorem. The nilpotent variety intersected with $S$ is equal to $\chi^{-1}(0, 0, 0)$, the surface singularity defined by $X^4 - YZ$.

## 8.7 A resolution of the adjoint quotient

To conclude the discussion of Brieskorn's theorem let us now quickly look at the other strategy available to identify the singularity. It uses the resolution of the unipotent variety by Springer. This gives a minimal resolution of the singularity and by showing that this resolution has the properties it should have, see proposition 3, the identification follows. For details see [32]. There is a recent review of the theory of conjugacy classes in semisimple algebraic groups by Humphreys [14].

Let $G, T, B, \mathbf{g}, \mathbf{h}$ be as before. Analogously to the adjoint quotient for the Lie algebra there is an adjoint quotient for the algebraic group $G$. There is an isomorphism $\mathbb{C}[G]^G \to \mathbb{C}[T]^{\mathcal{W}}$. The algebras of invariants $\mathbb{C}[G]^G$ and $\mathbb{C}[T]^W$, are freely generated as $\mathbb{C}$-algebras by the fundamental characters $\chi_i := \chi_{\lambda_i}$, and their restrictions $\chi_{i|T}$ to $T$ respectively.

The adjoint quotient for $G$ is then the mapping

$$\begin{aligned} \chi : G &\to T/\mathcal{W} \\ x &\mapsto (\chi_1(x), ..., \chi_r(x)), \end{aligned}$$

realized by the fundamental characters of $G$.

For all $f \in \mathbb{C}[G]^G$ we have $f(x) = f(x_s)$, so the unipotent variety $U$, is the fiber above $e$, the semisimple part of a unipotent element.

Now let $\mathcal{B}$ be the projective variety of Borel groups, which we can identify with the quotient $G/B$ by $G/B \to \mathcal{B}$, $gB \mapsto gBg^{-1}$. Take the subvariety of $\mathcal{B} \times G$,

$$\mathcal{C} := \{(B, x) \in \mathcal{B} \times G | x \in B\}.$$

We have natural morphisms $\tau : \mathcal{C} \to T$, $\pi : \mathcal{C} \to G$, such that we get the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \to & G \\ \downarrow & & \downarrow \\ T & \to & T_{/\mathcal{W}} \end{array}$$

**Theorem 8.10.** *The diagram above is a resolution of the adjoint quotient. In particular for each $t \in T$ the induced map*

$$\pi : \tau^{-1}(t) \longrightarrow \chi^{-1}(\bar{t})$$

*is a resolution of singularities of the fiber of $t$.*

A special case of this theorem, conjectured by Grothendieck, is Springer's resolution of the unipotent variety $U$:

$$\pi : \mathcal{C}_U \to U.$$

Here we take $\mathcal{C}_U$ the subvariety of $\mathcal{B} \times G$,

$$\mathcal{C}_U := \{(B, x) \in \mathcal{B} \times G | x \in B, x \in U\}.$$

We can identify $\mathcal{C}_U$ with the associated bundle $G \times^B U$, a smooth irreducible variety of dimension $\dim(G) - \mathrm{rank}(G)$, by the morphism $\tau : G \times^B U \to \mathcal{C}_U$, $g * u \mapsto (gB, gug^{-1})$. Now the morphism $\pi$ factors through the second projection $p : \mathcal{B} \times U \to U$, so $\pi = p \circ \tau$ is proper.

We have the following result on regular unipotent elements in $G$.

**Proposition 8.11.** *Let $x \in G$ be unipotent. Then the following are equivalent:*

- *$x$ is regular, dim($C_G(x)$)= $r$.*
- *$x$ is contained in a unique Borel subgroups of $G$.*
- *The adjoint quotient $\chi : G \to T/W$ is regular, i.e. smooth at $x$.*

*Proof.* For a proof see [14]. □

So onto the regular points of $U$, $\pi$ is one to one. The variety $U$ is the closure of the regular orbit, the smooth points of $U$. This concludes the proof of the claim that $\pi$ is a resolution of the singularities of $U$.

The way to proceed now is to proof that this resolution is of the desired form. It can be shown that this is in fact the case, namely the exceptional set of a subregular nilpotent element is a union of projective lines which intersect transversally and have self-intersection number $-2$. They have the Dynkin diagram corresponding to $G$ as a resolution graph.

# 9 Conclusion

Let us try to summarize the connections we have discussed. We have seen three theories that fall into the $ADE$ classification pattern. We can make the following table:

| | Theory | Nodes | Matrices |
|---|---|---|---|
| I | Finite subgroup $SU_2(\mathbb{C})$ | Irreducible representations | decomposition coefficients $a_{ij}$ |
| II | Lie algebra of type ADE | Simple roots | Cartan matrix |
| III | Minimal resolution of $\mathbb{C}^2/\Gamma$ | Components $E_i$ of the exceptional set | Intersection coefficients |

Table 8: The $ADE$ correspondence.

First of all there is a structural correspondence between these theories. They have a classification of the same type. We have shown that by computations within these theories, it is possible to relate the objects and matrices listed above. See the characterization of Kleinian singularities in proposition

5.1, the McKay correspondence in theorem 6.1, and the Geometric McKay correspondence in theorem 7.1.

Secondly there are direct links between the theories, one of the most remarkable the fact that the subregular singularity in the Lie algebra is exactly the quotient singularity corresponding to this Lie algebra.

There is a generalization of the McKay correspondence to higher dimensions. For a recent review see [29]. The picture we sketched in the introduction is then generalized to arbitrary dimension $n$:

$$\begin{array}{cc} & \mathbb{C}^n \\ & \downarrow \pi \\ \tilde{S} \rightarrow & \mathbb{C}^n/\Gamma. \end{array}$$

Here $\Gamma$ is a finite subgroup of $SL_n(\mathbb{C})$, such that the Gorenstein singularity $\mathbb{C}^n/\Gamma$ allows a crepant resolution. For $n = 3$ it is proven that there is a McKay correspondence. The first indication for this came in fact from the work of string theorists, who proved that the Euler number of $\tilde{S}$ equals the number of conjugacy classes in $\Gamma$.

For recent articles on generalizations of the McKay correspondence and related topics see for instance: www.maths.warwick.ac.uk/ miles/McKay/.

I want to finish here with a 'second-hand' citation of Proust, that I took from [13], and which presents a very nice way to look at the theory discussed in my thesis:

> '...si les choses se répètent, c'est avec de grandes variations.'
> Proust, La Prisonnièrre

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